N/D 3 2011 You can't change the bowl in mid-stream

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Last fall's N/D3 problem was about filling a bowl with water while evaporation took place. Under the stated conditions, could the bowl fill? It could not. Last spring one of your readers suggested that the calculations might be avoidable. If the bowl filled it would be filled indefinitely into the future. By a kind of "time reversal" from classical physics the same would hold indefinitely into the past. Since was not the case, the bowl would never fill. This seems wrong. The answer did depend on the shape of the bowl. If shaped differently it could fill and yet not overflow.

The work here is simplified by forcing virtually all constants to be 1. Let $f(h)$, $0 \le h \le 1$, be the cross sectional area of the bowl as a function of height. This is the only information about the bowl used. Without evaporation, the change in the volume $V(t)$ is linear:

$$\frac{dV(t)}{dt} = f(h(t))\,\frac{dh}{dt} = 1, \text{ or } \frac{dh}{dt} = f(h(t))^{-1}.$$

Evaporation is proportional to $f(h(t))$: $\frac{dV(t)}{dt} = f(h(t))^{-1} - f(h(t))$

In the original problem we used our knowledge of $f(h)$ to solve for $h(t)$. Here, the function $h(t)$ is proposed, "reverse engineering" determines $f(h)$. Circular symmetry is nice but not necessary. Continuity of $h(t)$ and its derivative are reasonable in order to keep the mathematics physically realizable. The proposal is that:

$h(t) = 2t - t^2$ (so $t(h) = 1 - (1 - h)^{1/2}$) and then $\frac{dh}{dt}(t) = 2 - 2t$ for $0 \le t \le 1$,

$h(t) = 1$ and then $\frac{dh}{dt}(t) = 0$ for $1 \le t$.

The equation for $f(h(t))$ is: $2 - 2t = f(h(t))^{-1} - f(h(t))$ or $f(h(t))^2 + (2 - 2t)\, f(h(t)) - 1 = 0$

Solving gives: $f(h(t)) = t - 1 \pm (t^2 - 2t + 2)^{1/2}$. The sign is negative. $f(0) = \sqrt{2} - 1$, $f(1) = 1$.

The function f can be expressed with h as the independent variable by solving: $f(h)^{-1} - f(h) = 2 - 2t(h) = 2(1 - h)^{1/2}$ to get the shape of the bowl, Implicit differentiation shows that $\frac{df}{dh} \to \infty$,as $h \to 1$. The bowl widens and the brim flares out. This is a flower bowl.

There are no difficult mathematical singularities at $t = h = 1$. All the functions of the solution extend past 1. The problem is physical. The bowl doesn't change. It was constructed using a negative sign in the definition of $f(h(t))$ but f is really a two valued function of t. Past 1 the sign has to be positive for $h(t)$ to remain a parabola. To do this we would need some immense quantum flux to instantly change the bowl. Even if this were to happen it is not clear what would start this extension down the parabola as opposed to everything remaining fixed as is now the case.

Higher derivatives don't exist but can be insured by using $h(t) = 1 - \exp(1-(h-1)^{-2})$ instead. This h is a slight variation of the canonical example of function in C^∞ but not analytic. The calculations look terrible.