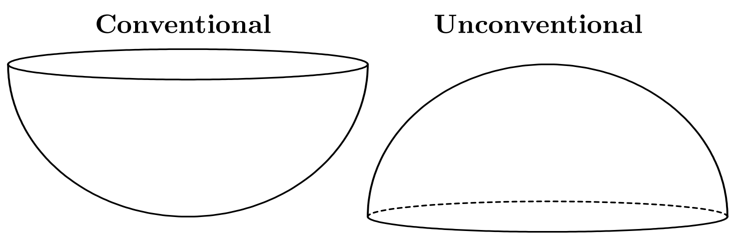
**Problem N/D 3 – Technology Review, November/December issue 2011**
Burgess H Rhodes, XVIII, 1960

**The Problem.**

> "Ermanno Signorelli has an empty bowl with a hemispherical concavity. The plane of its lip is horizontal and the environment (temperature, pressure, humidity, etc.) is constant. A liquid enters the bowl at a rate of $r$ m$^3$/sec (cubic meters per second) and evaporates at a rate of $ea$ m$^3$/sec, where $a$ is the current surface area in meters squared (assuming no meniscus). We can adjust $r$, but $e$ is fixed.
>
> Suppose $r$ is chosen so that when the bowl is full, the evaporation just matches the fill rate, and hence the bowl remains full. What is the internal diameter of the bowl?
>
> Using this same value for $r$, how long will it take to fill an initially empty bowl?"

**Logical Analysis.** With parameter constants **fill rate** $r$ (volume per unit time) and **evaporation rate** $e$ (volume per unit surface area per unit time), time to fill the hemispherical container should be the same, whether the container is "right side up" (conventional) or "upside down" (unconventional). The total volumes are the same and, for these two orientations, the cross section areas are the same, though the orders of the cross sections are reversed. If the radius of the bowl is $R$, then the liquid surface area is $S(R) = \pi R^2$ when the conventional container is full. The evaporation rate and fill rate are then equal, or $r = e \cdot S(R)$.




Envision the liquid entering the container through an orifice at the bottom. Liquid entering at rate $r$ into the unconventional container wets the bottom of the container whose surface area is $S(R) = \pi R^2$. There the rate at which liquid evaporates is equal to the rate at which it enters, so the container remains empty. Time to fill the container is infinite, and the container cannot be filled. This holds for both orientations, the difference being that some liquid accumulates in the conventional container. A mathematical confirmation for the conventional container follows.

**Basic Formulas.** Represent the bowl as the surface generated by revolving about the $y$-axis the bottom half of the circle $x^2+(y-R)^2 = R^2$, in which $R$ is the radius of the circle (and hence bowl). With $h$ denoting the height of liquid in the bowl, the volume of liquid is (use "horizontal washers")

$$V(h) = \int_0^h \pi(R^2 - (y-R)^2)dy = \pi(Rh^2 - \frac{h^3}{3})$$

and the liquid surface area is

$$S(h) = \frac{dV(h)}{dh} = \pi(2Rh - h^2). \tag{1}$$

**Diameter of the Bowl.** When the bowl is full, $h = R$ and $r = e \cdot S(R) = e \cdot \pi R^2$ (fill rate = evaporation rate). The **internal diameter** $D$ of the bowl is

$$D = 2R = 2\sqrt{\frac{r}{e\pi}}.$$

**Time to Fill the Bowl.** Let function $h(t)$ be the height of liquid in the bowl at time $t$. Initially $h(0) = 0$. Then

$$\frac{dV(h(t))}{dt} = r - e \cdot S(h(t))$$

$$\frac{dV(h(t))}{dh}\frac{dh(t)}{dt} = r - e \cdot S(h(t)) \quad \text{[chain rule]}$$

$$S(h(t))\frac{dh(t)}{dt} = r - e \cdot S(h(t)) \quad \text{[(1) above], and}$$

$$\frac{dh(t)}{dt} = \frac{r - e \cdot S(h(t))}{S(h(t))}.$$

The last differential equation is separable and, with $r = e \cdot S(R)$, leads to

$$\int \frac{S(h)}{S(R) - S(h)} dh = e \int dt$$

or, with (1) above,

$$\int \frac{2Rh - h^2}{(R-h)^2} dh = e \int dt. \tag{2}$$

With a little integration, and with initial condition $h(0) = 0$, equation (2) yields

$$\frac{h^2}{R-h} = e \cdot t \quad \text{or} \quad t(h) = \frac{1}{e}\frac{h^2}{R-h}.$$

As liquid enters the bowl, $V(h)$ increases because for $h < R$, $S(h) < S(R)$ so that $r > e \cdot S(h)$. Thus $h$ increases, and time $T_o$ to fill is

$$T_o = \lim_{h \to R^-} t(h) = \lim_{h \to R^-} \frac{1}{e}\frac{h^2}{R-h} = \infty.$$

Time to fill the hemispherical bowl is **infinite**, and the bowl will never be filled! It makes no difference that the bowl started out empty.

**Generalization.** Consider any container $\mathcal{C}$ which can hold a liquid. If there is already liquid in $\mathcal{C}$, height $h = 0$ at the current liquid level, and $h$ measures height above the current liquid level. Denote by $V(h)$ and $S(h)$ the volume and (top) surface area of the liquid when the height of the liquid in $\mathcal{C}$ is $h (\geq 0)$. Suppose liquid enters $\mathcal{C}$ at fixed rate $r$ (volume per unit time), and liquid evaporates at the rate $e \cdot S(h)$, with $e$ a fixed rate (volume per unit area per unit time). Thus

$$\frac{dV(h(t))}{dt} = r - e \cdot S(h(t)). \tag{3}$$

Assume that $S(h)$ is a continuous function of $h$. Thus there are no “ledges” in the wall of the container. $\mathcal{C}$ is initially empty (condition $V(0) = 0$). If $e \cdot S(0) > r$, no liquid will accumulate in $\mathcal{C}$. If $r > e \cdot S(0)$, liquid level will continue to increase if no $h$ is encountered for which $e \cdot S(h) = r$.

Let $h = H_o$ be the smallest height for which $e \cdot S(H_o) = r$. Then if $h = H_o$ the volume of liquid in $\mathcal{C}$ remains constant. If there is no such $H_o$, then $\mathcal{C}$ will fill (overflow).

Introduce the condition $r = e \cdot S(H_o)$ into (3):

$$\frac{dV(h(t))}{dh}\frac{dh(t)}{dt} = S(h(t))\frac{dh(t)}{dt} = e \cdot S(H_o) - e \cdot S(h(t)),$$

$$\frac{dh(t)}{dt} = \frac{e \cdot S(H_o) - e \cdot S(h(t))}{S(h(t))}, \quad \text{and}$$

$$\int \frac{S(h)}{S(H_o) - S(h)} dh = e \int dt = e \cdot t(h) \tag{4}$$

determines the time $t(h)$ it takes for the liquid level to reach height $h$.

To evaluate (4) let $u = S(H_o) - S(h)$ and assume that $S$ is differentiable. Then $du = -S'(h)dh$ and equation (4) is transformed to

$$-\int \frac{S(H_o) - u}{u} \frac{1}{S'(h)} du = e \cdot t(h). \tag{5}$$

Suppose $S'(h)$ is bounded by $0 < m \leq S'(h)$ so that liquid surface area is increasing with respect to $h$ at no less a rate than $m$. With no loss of generality, adjust the definition of $h$ so that $0 < m \leq S'(h)$ for $0 \leq h \leq H_o$. Then

$$\begin{aligned} e \cdot t(h) = -\int \frac{S(H_o) - u}{u} \frac{1}{S'(h)} du &\geq -\frac{1}{m} \int \frac{S(H_o) - u}{u} du \\ &= -\frac{1}{m}[S(H_o) \ln u - u] + C \\ &= -\frac{1}{m}[S(H_o) \ln(S(H_o) - S(h)) - (S(H_o) - S(h))] + C. \end{aligned} \tag{6}$$

Constant $C$ is determined so that $t(0) = 0$:

$$C = \frac{1}{m}[S(H_o) \ln(S(H_o) - S(0)) - (S(H_o) - S(0))]. \tag{7}$$

Equations (4), (6), and (7) together yield

$$t(h) \geq \frac{1}{e} \frac{1}{m}[S(H_o) \ln\left(\frac{S(H_o) - S(0)}{S(H_o) - S(h)}\right) - (S(h) - S(0))].$$

The time $T_o$ for liquid to reach height $H_o$ is then

$$T_o = \lim_{h \to H_o^-} t(h) \geq \frac{1}{e} \frac{1}{m} \lim_{h \to H_o^-} [S(H_o) \ln\left(\frac{S(H_o) - S(0)}{S(H_o) - S(h)}\right) - (S(h) - S(0))] = \infty.$$

This result continues to hold as $m \to 0^+$. It is impossible to fill a container to a height $H_o$ at which r= $e \cdot S(H_o)$ and $S'(h) > 0$ for $h$ immediately below $H_o$.

**Conclusion.** By the initial **Logical Analysis** and the **Generalization**, for a container of **any shape**, if there is a horizontal cross section at liquid height $h = H_o$ with area $S(H_o)$ and for which $r = e \cdot S(H_o)$, it is impossible to fill the container to height $H_o$. Having ignored all environmental and physical considerations, as in the problem statement, this is of course a strictly mathematical conclusion. If the container must be filled through height $H_o$, just increase $r$. Or in the case of the unconventional container, just tilt it for a moment to get the filling started.