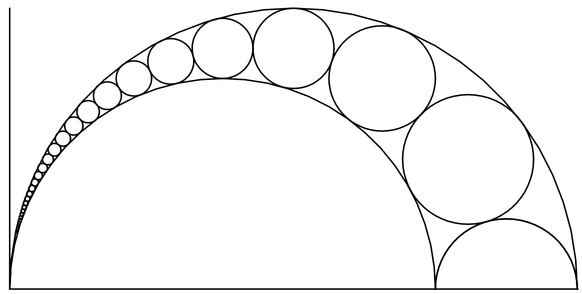
Geometric Inversion and the Pappus Chain

(Inspired by Technology Review problem 3, March/April issue 2011)

Burgess H Rhodes

Properties of the **Pappus Chain**, the geometric figure shown here, are explored with the aid of the geometric inversion transformation.

The Pappus Chain



Geometric Inversion*. Denote by $\mathcal{C}(x, y, r)$ a circle with center $O = (x, y)$ and radius r. If $\mathcal{C}_\mathcal{I}$=$\mathcal{C}(x_I, y_I, r_I)$ is used to define a **geometric inversion** $\mathcal{I}$:$R^2 \to R^2$, circle $\mathcal{C}_\mathcal{I}$ is called an **inversion circle** with **inversion center** (x_I, y_I) and **inversion radius** r_I.

Let $P = (x, y)$, $P' = (x', y')$, and denote the geometric inversion by $(x', y') = \mathcal{I}(x, y)$ or equivalently $P' = \mathcal{I}(P)$. In geometric terms, the image $P' = \mathcal{I}(P)$ is the point on the line through O_I and P for which

$$\frac{\overline{O_I P'}}{r_I} = \frac{r_I}{\overline{O_I P}}; \tag{1}$$

$\overline{O_I P}$ is the distance between O_I and P. In coordinate form, the geometric inversion is defined by

$$\begin{aligned} x' &= x_I + M(x - x_I) \\ y' &= y_I + M(y - y_I), \text{ with} \\ M &= \frac{r_I^2}{(x - x_I)^2 + (y - y_I)^2}. \end{aligned} \tag{2}$$

If $\mathcal{G}$ is a geometric figure, then the inversion $\mathcal{G}'$= $\mathcal{I}(\mathcal{G})$ of $\mathcal{G}$ is a geometric figure:

$$\mathcal{G}' = \{(x', y') \mid (x', y') = \mathcal{I}(x, y), (x, y) \in \mathcal{G}\}. \tag{3}$$

Circles and lines (think "circles of infinite radius") are transformed under $\mathcal{I}$ into circles and lines. Lines through O_I are invariant, and $\mathcal{C}_\mathcal{I}$ is invariant.

If $\mathcal{G}$ =$\mathcal{C}(x_a, y_a, r_a)$, then $\mathcal{G}'$=$\mathcal{I}(\mathcal{G})$=$\mathcal{C}(x'_a, y'_a, r'_a)$, where

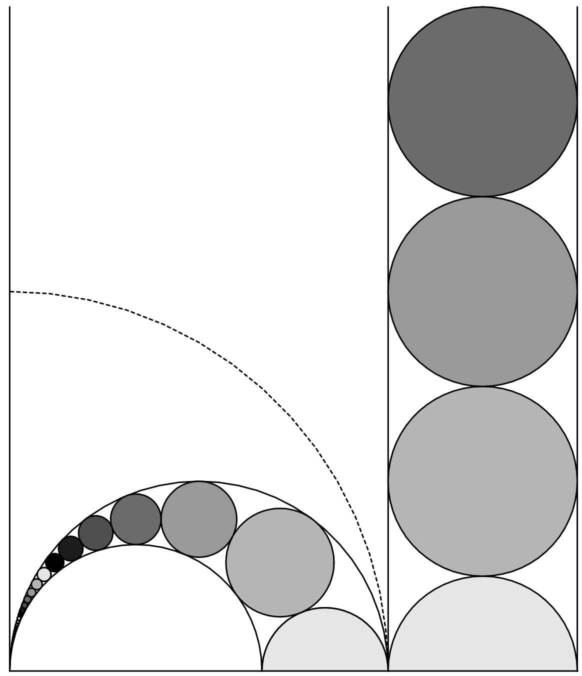
$$\begin{aligned} r'_a &= |S| r_a \\ x'_a &= x_I + S(x_a - x_I) \\ y'_a &= y_I + S(y_a - y_I), \text{ with} \\ S &= \frac{r_I^2}{(x_a - x_I)^2 + (y_a - y_I)^2 - r_a^2}. \end{aligned} \tag{4}$$

Note that $(x'_a, y'_a) \neq \mathcal{I}(x_a, y_a)$; center (x'_a, y'_a) is not the image of center (x_a, y_a), unless $r_a = 0$ (compare equations (2) and (4)).

* Adapted from *http://mathworld.wolfram.com/Inversion.html*

Pappus Chain. The Pappus Chain can be obtained as a geometric inversion of a column of circles with equal radii ρ and with two vertical tangent lines on either side of the column, shown in the following figure. The center O_I is on a horizontal line passing through the center of one of the stacked circles, and O_I may be chosen so that the (dotted) circumference of $\mathcal{C}_{\mathcal{I}}$ is tangent to a vertical tangent.

The Pappus Chain Obtained by Geometric Inversion



With no loss of generality, choose $\mathcal{C}_{\mathcal{I}}$=$\mathcal{C}(0,0,1)$. Then the image $\mathcal{I}$(left vertical tangent) is the upper bounding semi-circle $\mathcal{C}_U$ =$\mathcal{C}(\frac{1}{2},0,\frac{1}{2})$. The image $\mathcal{I}$(right vertical tangent) is the lower bounding semi-circle $\mathcal{C}_L$=$\mathcal{C}(\frac{1}{2(1+2\rho)},0,\frac{1}{2(1+2\rho)})$. The stacked circles, from bottom upward, transform into the Pappus Chain circles in the counterclockwise direction. In the stack of circles, the points of tangency between adjacent circles all lie on a vertical line whose image is a semi-circle between the bounding semi-circles. Thus the points of tangency of the Pappus circles lie on a circle.

Denote the radii of the upper and lower bounding semi-circles by r_U and r_L, and their centers by O_U and O_L. If $O_c=(x_c,y_c)$ and r_c are the center and radius of a Pappus circle or any circle inscribed between the bounding semi-circles, then $\overline{O_cO_U}+\overline{O_cO_L}=(r_U-r_c)+(r_L+r_c)=r_U+r_L$, a constant. Consequently the centers of the Pappus circles lie on an ellipse with foci O_U and O_L.

The k^{th} stacked circle is $\mathcal{C}_k$=$\mathcal{C}(x_k,y_k,r_k)=\mathcal{C}(1+\rho,2k\rho,\rho),\ k=0,1,2,\ldots$

With $\mathcal{C}_{\mathcal{I}}$=$\mathcal{C}(0,0,1)$, equations (4) specialize to

$$\begin{aligned} r'_a &= |S|r_a \\ x'_a &= Sx_a \\ y'_a &= Sy_a,\ \text{and} \\ S &= \frac{1}{(x_a)^2+(y_a)^2-r_a^2}. \end{aligned} \tag{5}$$

Consequently in the Pappus Chain, $S > 0$ and for the k^{th} stacked circle, $k = 0, 1, 2, \ldots$

$$\begin{aligned}
r'_k &= \frac{r_k}{(x_k)^2 + (y_k)^2 - r_k^2} = \frac{\rho}{(1+\rho)^2 + (2k\rho)^2 - \rho^2} = \frac{\rho}{4k^2\rho^2 + 2\rho + 1}, \\
\frac{x'_k}{r'_k} &= \frac{Sx_k}{|S|r_k} = \frac{S(1+\rho)}{S\rho}, \quad \text{or} \quad x'_k = \frac{1+\rho}{\rho} r'_k = \frac{1+\rho}{4k^2\rho^2 + 2\rho + 1}, \text{ and} \\
\frac{y'_k}{r'_k} &= \frac{Sy_k}{|S|r_k} = \frac{2kS\rho}{S\rho} = 2k, \quad \text{or} \quad y'_k = 2kr'_k = \frac{2k\rho}{4k^2\rho^2 + 2\rho + 1}.
\end{aligned} \tag{6}$$

The k^{th} Pappus Chain circle is $\mathcal{C}'_k = \mathcal{C}\left(\frac{1}{4k^2\rho^2 + 2\rho + 1}(1+\rho, 2k\rho, \rho)\right)$. From $y'_k = 2kr'_k$ we see that the distance y'_k that the center of the k^{th} Pappus circle is above the horizontal diameters of the boundings semi-circles is k times the diameter $2r'_k$ of the k^{th} Pappus circle.