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RE: Puzzle Corner, Technology Review, Problem J/A 2

J/A 2. $n \geq 1$ hats belonging to n people are mixed up in a cloak room. Let a_n be the number of permutations of hats in which nobody gets the right hat. A corollary of the definition of a_n is the recursion relation:

$$a_n = n! - \sum_{k=1}^{n} \binom{n}{k} a_{n-k} \tag{1}$$

since a_n equals the total number of permutations less the number of permutations in which exactly $k = 1,2,\ldots,n$ people get the right hat. We introduced the notation $a_0 = 1$ in (1) so that for one hat, one owner, $a_1 = 0$.

The probability that nobody gets the right hat is $p_n = a_n/n!$ if all $n!$ permutations are equally likely. Using the notation $p_0 = 1$, and dividing both sides of (1) by $n!$, we obtain:

$$p_n = 1 - \sum_{k=1}^{n} \frac{p_{n-k}}{k!} \tag{2}$$

Recursion relations (1-2) are sufficient to calculate a_n and p_n as shown in the table below:

n = number of hats	a_n =number of permutations in which no hat is right	$p_n = a_n/n!$ = probability that no hat is right
1	0	0
2	1	1/2
3	2	1/3
4	9	3/8
5	44	11/30
6	265	53/144
…		…
n→ ∞		e^{-1}

Let $p = \lim_{n\to\infty} p_n$. It follows from the recursion relation (2) that:

$$p = 1 - p\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) = 1 - (e-1)p \;=\; e^{-1} \tag{3}$$

The solution to the recursion relation (2) is the n-term partial sum in the Taylor expansion of e^{-1} :

$$p_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \tag{4}$$

To obtain this solution, we define $d_0 = 1$; $d_n = p_n - p_{n-1}$ for $n > 0$, and substitute the expansions of p_n and p_{n-1} from (2). This yields:

$$0 = d_0 + \frac{n!}{(n-1)!} d_1 + \frac{n!}{(n-2)!} d_2 + \cdots + \frac{n!}{2!} d_{n-2} + \frac{n!}{1!} d_{n-1} + n!\, d_n \tag{5}$$

It is readily seen that (5) is satisfied by $d_k = (-1)^k / k!$ by using the binomial expansion:

$$0 = (1-1)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{(-1)^k}{k!} \tag{6}$$

The probability p_n is obtained by summing the differences: $p_n = \sum_{k=0}^{n} (p_k - p_{k-1}) = \sum_{k=0}^{n} d_k$ from which (4) follows.