Puzzle Corner, July/August 2010-2, 8x8 Sudoku-like problem, cover letter

Hi Allan,

Following is my solution to J/A 2 2010.

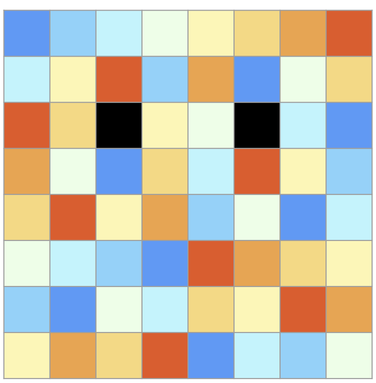
The puzzle asks if it is possible to fill in an 8x8 grid with digits 1-8 so that no digit repeats in any row, column or non-wrapping diagonal leaving at most one cell blank. Based on a complete computer search, the answer is no.

My algorithm starts by finding all valid ways 8 copies of one digit can be placed. (This is the same as the classic 8 queens problem, with 92 solutions.) It then successively finds all ways to add 8 valid copies of a new digit without overlap to all current patterns. When tried, this process becomes impossible when trying to include the seventh digit. Since any solution to the original problem with one or no blanks would have seven of the eight digits fully assigned, there are no such solutions.

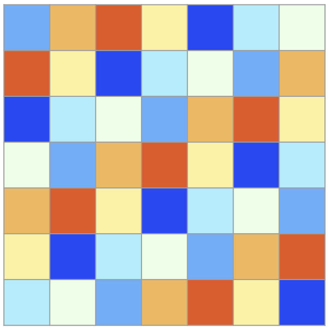
Attached is a PDF file with the Mathematica code that does this computation. By only keeping one copy of each potential intermediate solution out of all equivalent ones under symmetries of the square or permutations of the digits, the number of cases to check stays small and the code is fast.

Although not part of the problem, I searched for all solutions with 2 blanks. Using the above, these must have 6 digits with 8 instances and 2 with 7. The algorithm is the same as above for the first six digits. For the last two, the patterns to check come from the broader class of all ways 7 copies of one digit can be validly placed. After selecting a unique representative from each symmetry/permutation class, there are 12 such solutions, which are listed in the PDF file.

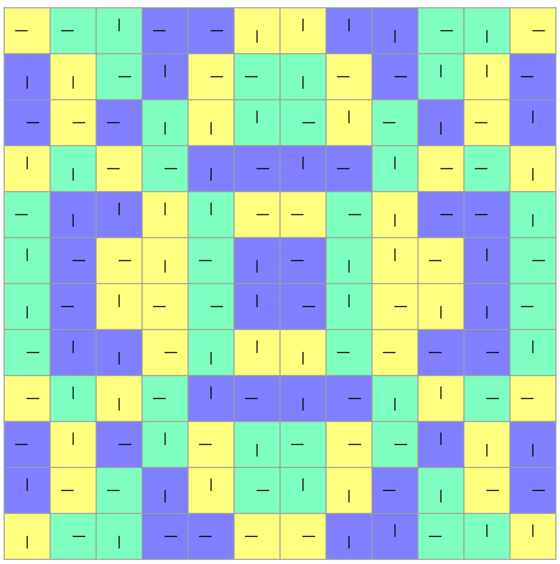
As an example, here is one of them with colors for digits and black for blanks:



Following are some other things I learned while thinking about the problem; and gradually improving (and complicating) my algorithm. For a 9x9 grid there are no solutions with two blanks, but many with three. There are no perfect 10x10 solutions. For any grid dimension not divisible by 2 or 3 there are always perfect solutions, which work even if diagonals are allowed to wrap. (This result goes back to at least George Pólya in 1918.) For example, here is one for 7x7:

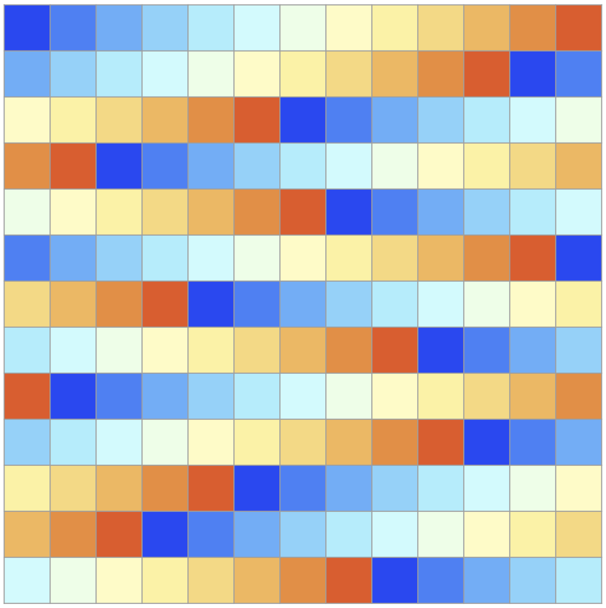


There are similar looking solutions for 11, 13, 17,... . In contrast, here is a solution for 12x12:

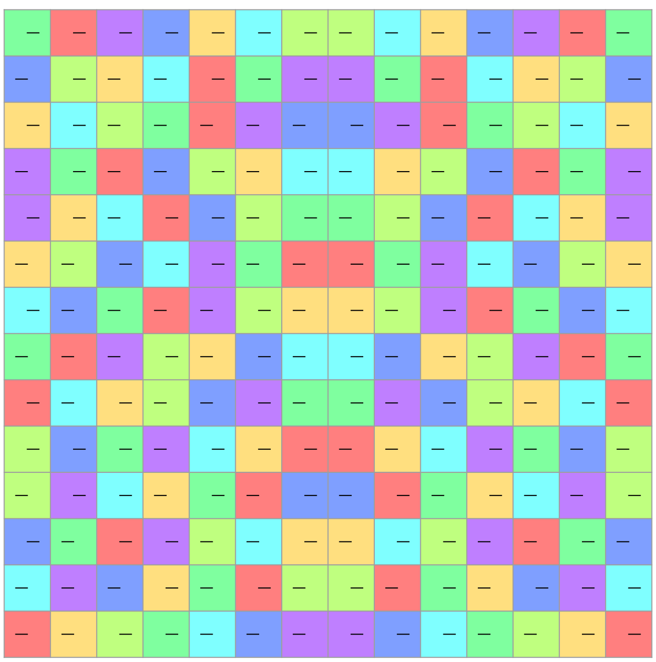


This one is colored to show its quasi-symmetries. Each of the three colors is a left-right, up-down symmetrical pattern. Each color pattern is composed of the four left-right and up-down reflections of one queen pattern. The position of the tic marks is constant for each of the four queen patterns in one color.

Here is a solution for 13x13 that is based on shifting one wrapping queen pattern successively to the right. It is less regular than one formed like the 7x7 above.



Here is a 14x14 solution:



Each of the seven colors is a left-right symmetrical pattern, composed of the left-right reflections of one queens pattern.

It is known that there are non-wrapping queens solutions for 4x4 and larger grids. For example, see http://web.telecom.cz/vaclav.kotesovec/rivin_1994.pdf for a simple construction. In fact, for grid size not divisible by 2 or 3, there are wrapping (toroidal) solutions, which all extend to grid filling patterns by shifting.

For grid sizes divisible by 2 or 3, I could not find much about grid filling patterns on the Internet. Based on generating examples, it seems extremely likely that there are such patterns for all sizes greater than or equal to 12. It would be nice to have a construction for at least one of each size.

There is a whole industry under the heading of Latin Squares devoted to counting all possible patterns subject to various conditions. In general, these numbers grow faster than exponentially with grid size but frustrating little is known. It is all computationally quite demanding as the dimension grows and these problems are usually NP-complete.

Thank you again for the puzzle corner.

Joel
7/16/10
modified 8/24/10