Technology Review Puzzle Corner
Proposed Answer: Sept/Oct 2009 problem 3
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We are asked to find the probability, P, that a tennis game will be won, given the probability, p, of winning each point. To win a game, at least four points, and at least two more than the opponent, must be won.

For S_4, S_5 and S_6 the probability of winning, W, (or losing, L,) the game on that point is the probability of having won (or lost) exactly three of the preceding points and the current point.

Point	W	L
S_4	p^4	$(1-p)^4$
S_5	$4p^4(1-p)$	$4p(1-p)^4$
S_6	$10p^4(1-p)^2$	$10p^2(1-p)^4$

Note: For W_5 and L_5 there are four possible scenarios; for W_6 and L_6 there are 10.

If after six points the game has not ended, the game must be tied at 3-3 (deuce). Thereafter, the game cannot end on an odd-numbered point and can be won (or lost) only by winning (or losing) a pair, T, of consecutive odd-even points ($S_{7\text{-}8}$, $S_{9\text{-}10}$, …). Thus, after the first six points there follows an infinite series of point pairs, each of which is progressively less likely to be played. For each pair of points T_i, the contribution to the probability of winning (W_i) or losing (L_i) the game is equal to the probability of winning or losing both points (p^2 and $(1-p)^2$, respectively) times the probability that the pair of points will be played (i.e., the probability that the game is not yet over, Q_i).

The probability that the first pair of points, T_0 ($S_{7\text{-}8}$), will be played (Q_0) is equal to the probability that the game was not over after the 6^{th} point:

$$Q_0 = 1 - W_4 - L_4 - W_5 - L_5 - W_6 - L_6$$

For subsequent pairs, $T_{i=1\rightarrow\infty}$, the probability that the pair will be played is progressively reduced by the probability that the game was won on a prior pair, according to the formula

$$Q_i \;=\; Q_0 * (1 - p^2 - (1-p)^2\,)^{\,i}$$

The corresponding probability that the game will be won on any pair of pair of points T_i, is equal to

$W_i = p^2 * Q_i$.

For the infinite series of pairs, W_i converges and the sum of the series is equal to

$$\Sigma_{i=0\rightarrow\infty} W_i = p^2 * Q_0 / (p^2 + (1-p)^2).$$

Thus, the probability of winning the game is equal to

$$P = W_4 + W_5 + W_6 + \Sigma_{i=0\rightarrow\infty} W_i,$$

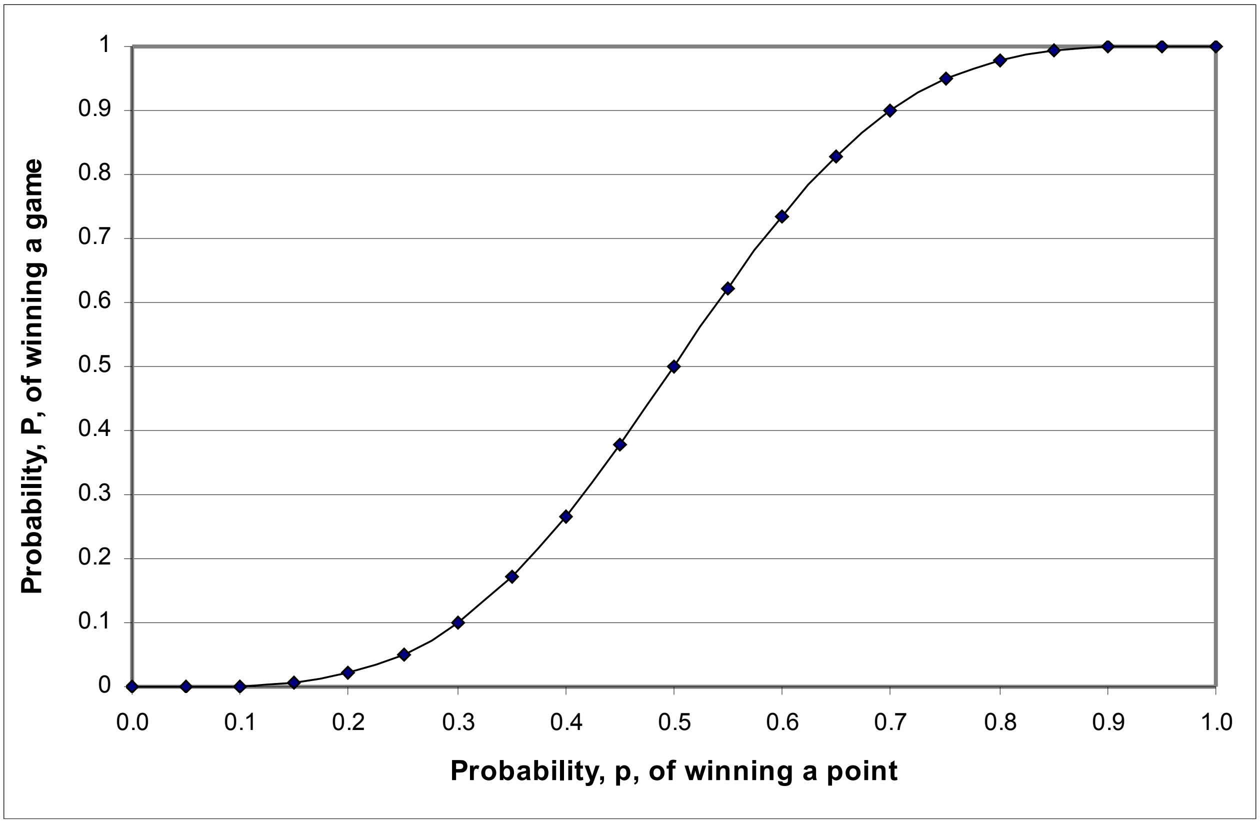
or explicitly,

$$P = p^4 + 4p^4(1-p) + 10p^4(1-p)^2 + p^2(1-p^4 - 4p^4(1-p) - 10p^4(1-p)^2 - (1-p)^4 - 4p(1-p)^4 - 10p^2(1-p)^4)/(p^2 + (1-p)^2).$$

A more computationally convenient form is

$$P = \frac{-8p^7 + 28p^6 - 34p^5 + 15p^4}{2p^2 - 2p + 1}$$

The results are depicted graphically here:


Probability, P, of winning a game
Probability, p, of winning a point
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0