ANSWER TO PUZZLE M/A 2 BY AVI ORNSTEIN
SUBMITTED BY ROBERT ACKERBERG '56

The two sequences x_n and y_n satisfy the same linear second order difference equation below (*) but with different initial conditions, $x_1 = 1$, $x_2 = a$; $y_1 = 1$, $y_2 = (a - 1)$. Since the determinant

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = -1 \neq 0$$

the two sequences are linearly independent.
The simplest way to find these sequences is to express them in terms of two linearly independent solutions, w_n and z_n, which satisfy the same difference equation (*), with the following initial conditions: $w_1 = 1$, $w_2 = 0$; and $z_1 = 0$, $z_2 = 1$. Once these sequences have been determined, the unknown sequences may be written, using their initial conditions, as:

(1) $y_n = w_n + (a - 1)z_n$ and (2) $x_n = w_n + az_n$

By subtracting (2) from (1) we find

(3) $y_n = x_n - z_n$

which relates (answering Ornstein's request) the two sequences x_n and y_n in terms of the sequence z_n which satisfies

(*) $z_n = az_{n-1} - z_{n-2}$ with $z_1 = 0$ and $z_2 = 1$ as indicated above.

For the cases $a > 2$, [the case $a \equiv 2$ will be considered later] we assume a solution of the form $z_n = Ct^n$, where C is an arbitrary constant, and two values of t (h&k) are determined by solving a quadratic equation. Thus, we find

$$z_n = Ah^n + Bk^n$$

where $2h = a + b$, $2k = a - b$ and $b = \text{sqrt}(a^2 - 4)$. A and B are constants to be determined by applying the initial conditions. Carrying this out, we find

$$z_{n+1} = (1/b)[h^n - k^n] = (1/2^n)(1/b)\{(a+b)^n - (a-b)^n\}$$

After some algebra and replacing b with $\text{sqrt}(a^2-4)$, we can write this result in terms of the sums:

even $$z_{2(n+1)} = (1/2^{2n}) \sum_{s=0,1,2}^{n} [(2n+1)!/(2n-2s)!(2s+1)!]\, a^{2n-2s} (a^2 - 4)^{s} \qquad n=0,1,2\ldots\ldots$$

odd $$z_{2n+1} = (a/2^{2n-1}) \sum_{s=1,2,3}^{n} [(2n)!/(2n-2s+1)!(2s-1)!]\, a^{2n-2s} (a^2 - 4)^{s-1} \qquad n=1,2,3\ldots\ldots$$

Note the exceptional case when $a = 2$, for which $h \equiv k = 1$, i.e., only one solution for t is found. In this case the general solutions to (*) are of the form $(A + Bn)$ where A and B are arbitrary constants.
Applying the initial conditions we find

$$y_n = 1 \text{ and } x_n = n \quad \text{for } n = 1,2,3\ldots\ldots$$

Thus, $x_n = ny_n$ for the special case $a = 2$.