Solution to TR Puzzle 2007 S/O 3
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In what follows, I'll use the word *move* for a pass or a conclusion. Also, I assume the players are sitting so that the move passes clockwise, so B is to A's left and C to her right.

Each person knows that her number is either the sum or the (absolute value of the) difference of the other two numbers. Given that, how could A possibly know the answer after 0 moves? Only if $C = B$ (letting an italic capital letter be the numbers written on the corresponding person's hat). In that case A would have to be either 0 or 2. But it can't be 0, because all the numbers are positive integers, so it would have to be 2. But A *didn't* know the answer at the outset, so we can conclude that $C \neq B$.

Similarly, how could B know the answer after A's initial pass? If $C = A$, then she could do reasoning analogous to A's, so we can conclude that $C \neq A$. But in addition, we can infer that $A \neq 2C$, because if she saw that $A = 2C$ that would mean that either $B = C$ or $B = 3C$. But we know $B \neq C$ from A's pass, so B would be able to conclude $B = 3C$. But B concluded no such thing; therefore, $A \neq 2C$.

On C's initial-round move, we can similarly rotate the constraints around, yielding

$$B \neq A \text{ and } B \neq 2A \text{ and } \ldots$$

where the dots indicate what we get when we "rotate in" the information that $A \neq 2C$ the way we "rotated in" the information that $C \neq B$ on B's move. I'll fill in the dots in a moment. First, observe that the next time we rotate we won't get 4 new constraints, because the pattern on the left started with A, so it's stale news to her. It just goes away. That means that in general what happens is that

1. The second and third constraints from the previous move are translated merely by substituting letters cyclically, yielding the first and second new constraints.

2. The third new constraint is derived from the third old constraint by a method analogous to what we did above.

The method for deriving the third constraint in the new bunch is as follows: Let m, r, and l mean the numbers of the mover, the person to her right, and the person to her left, respectively, on the new move. We have an old constraint of the form $l \neq pm$, where p is 1 after move 1, and 2 after move 2. We assume (based on faith in the pattern so far) that the new constraint is also of the form $r \neq p'l$ (using the word "also" because the person labeled m now was labeled l on the previous move). So assume that the current mover, m, can derive the correct answer if $r = p'l$, and

$$p'l - l = \frac{1}{p}l$$

because then $m = p'l + l$ or $m = p'l - l$ (we will have to show that this is > 0 to justify leaving off the absolute-value symbols), and in the latter case, we would have

$$m = \frac{l}{p} \text{ and hence } l = pm$$

which is a no-no.

If we divide the equation

$$p'l - l = \frac{1}{p}l$$

by l, we can derive

$$p' = \frac{1}{p} + 1 = \frac{p+1}{p}$$

Obviously, if $p > 0$ then p' will be, justifying our omission of the absolute-value symbols above.

It may seem unnecessary to generalize, given that only 5 moves occur, but in fact the game could have gone on to an indefinite number of moves. (See below.) I guess the number 5 was chosen to make things interesting but still allow one to solve the problem by hand, although I don't recommend it.

After 5 moves, here are all the constraints we have:

1 A			$C \neq B$
2 B		$A \neq C$	$A \neq 2C$
3 C	$B \neq A$	$B \neq 2A$	$B \neq \frac{3}{2}A$
4 A	$C \neq 2B$	$C \neq \frac{3}{2}B$	$C \neq \frac{5}{3}B$
5 B	$A \neq \frac{3}{2}C$	$A \neq \frac{5}{3}C$	$A \neq \frac{8}{5}C$

In this table, constraints start in the third column and move left, permuting letters appropriately. The new third constraint has the same letters as the other two on its row, but, as proved above, we get its coefficient by flipping the fraction from the last constraint of the previous row over and adding its numerator and denominator to get the new numerator. So the numerators and denominators form staggered Fibonacci sequences.

On move 6, the pattern stops. Although we didn't prove that the *only* ways for the mover to be in a position to get the answer was for one of the three new constraints to fail, if one *does* fail then that would end the game. So we are looking for 3 numbers such that one is 72 and the others add up to 72, which satisfy all of the constraints above and in addition satisfy at least one of

$$6\ \mathrm{C} \quad B = \tfrac{5}{3}A \quad B = \tfrac{8}{5}A \quad B = \tfrac{13}{8}A$$

There is probably some deep mathematical insight that would allow one to immediately see what the answer must be, but because I am a Professor of Computer Science I wrote a program that just tries a bunch of numbers. (It will not win any prizes for efficiency in doing so, either, but today's computers are satisfyingly speedy. The program also generates all the constraints automatically, which saved me a bit of work and avoided a source of errors.)

The only triple that satisfies all the constraints and one of the equalities above is $A = 27, B = 45, C = 72$, which satisfies $B = \frac{5}{3}A$. That's my final answer.

Curiously, in case you're wondering whether there is something special about 144, the answer is No. It is not the first sum for which there is a solution. Here are all the smaller ones:

Sum	A	B	C
16	3	5	8
26	5	8	13
32	6	10	16
42	8	13	21
48	9	15	24
52	10	16	26
64	12	20	32
78	15	24	39
80	15	25	40
84	16	26	42
96	18	30	48
104	20	32	52
112	21	35	56
126	24	39	63
128	24	40	64
130	25	40	65
144	27	45	72

Note that it is always the case that $A < B < C$. I believe that the winner of the game will always turn out to have the biggest number on her hat, and the person to her left the smallest (after move 2), but to prove this I'd have to show that the only way to win is for one of the equalities of the form described above to be true. Assuming that's the case, it's pretty clear that whichever equality is chosen is of the form $R = pL$, with $p > 1$, which makes L smaller than R; then the style of reasoning parsed at the beginning would ensure that $M \neq R - L$, so M would have to $= R + L$.

It should be obvious that the game could end after any number of moves. The constraints rule out various ratios among the numbers, but there are always an infinite number of ratios left in the running (because there are an infinite number of prime numbers).

There is a tantalizing pattern to the table above. After the line with sum 42, every row is a multiple of a row already introduced. (Obviously, once a solution is introduced, all its multiples are solutions also.) The sum 144, for instance, is obtained from the second row by multiplying by 9. In fact, the row labeled 42 is the last *seed row*, one that is not a multiple of another. The three seed rows are sequential Fibonacci triples ($3 + 5 = 8$, $5 + 8 = 13$, $8 + 13 = 21$). The next triple in the sequence is $13 + 21 = 34$, but it isn't a solution. The number of seed rows is equal to the number of passes - 2 (the two startup moves?). By increasing the number of moves until the game ends, can you get more Fibonacci-triple rows? (Of course, the sequence probably start with a different first triple.) And will it always be the case that the table is produced by taking all multiples of some "seed" rows? A better number theorist than I will have to answer these questions, and besides I have spent too much of my supposedly valuable time on this!