0.2 2007 J/A 3

Phil Lally wants to know the area swept out horizontally by a 15π foot rope tethered to the outside of a 15 foot radius right circular cylinder. The area is $\frac{375}{2}\pi^3$ square feet. Unfortuantely, I can not demonstrate this without resorting to calculus.

The coordinates of the endpoint (x, y) of the tether in terms of the angle α of the point of last contact of the tether with the circle are:

$$\begin{aligned} x &= 15 - 15\cos\alpha + 15(\pi - \alpha)\sin\alpha \\ y &= 15\sin\alpha + 15(\pi - \alpha)\cos\alpha. \end{aligned} \tag{5}$$

The area A_1 of the upper right quadrant, in square feet, including the area of the semi-circle is

$$\begin{aligned} A_1 &= \int_0^{15\pi} dy\, x \\ &= \int_0^{\pi} d\alpha\, 15(\pi - \alpha)\, x(\alpha) \\ &= \int_0^{\pi} d\alpha\, 15(\pi - \alpha)(15 - 15\cos\alpha + 15(\pi - \alpha)\sin\alpha) \\ &= 225\left(\frac{1}{6}\pi^3 + \frac{1}{2}\pi\right) \end{aligned} \tag{6}$$

Above, $\frac{dy}{d\alpha} = -15(\pi - \alpha)\sin\alpha$ was used to change variables from y to α.

The area A_0 of the semi-circle without the silo in it is $A_0 = \frac{1}{2}\pi r^2 = \frac{225}{2}\pi^3$ ft^2. The area A_s of the silo is $A_s = \pi r^2 = 225\pi$ ft^2. The total area A is twice the area A_1 of the first quadrant plus the area of the empty semi-circle A_0, minus the area of the silo:

$$\begin{aligned} A &= 2A_1 + A_0 - A_s \\ &= 225\left(\frac{1}{3}\pi^3 + \pi\right) + 225\frac{1}{2}\pi^3 - 225\pi \\ &= \frac{375}{2}\pi^3. \end{aligned} \tag{7}$$

The total area is $\frac{375}{2}\pi^3$ ft^2, which is $\frac{5}{6}$ of the area of a circle with a 15π foot radius.

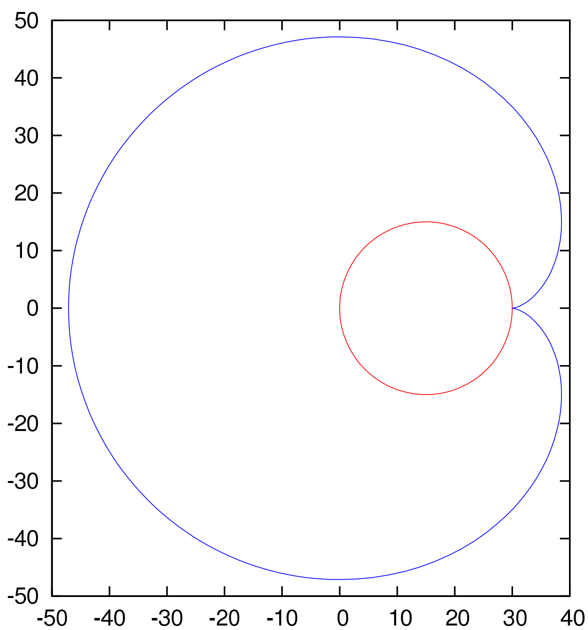

Figure 1: Boundary of area swept out by a 15 π foot rope tethered at the origin to a 15 foot radius circle.

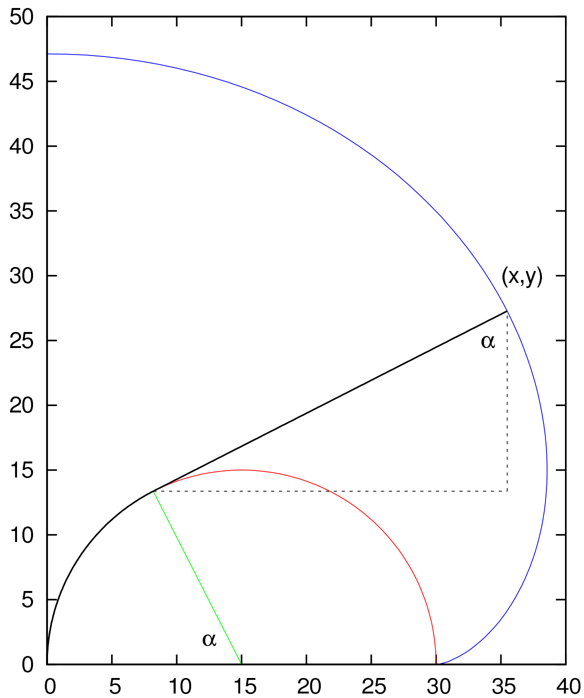

Figure 2: Detail of the coordinates used to calculate the area.