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Subject: July 2, Puzzle Corner

Problem: Avi Ornstein has circumscribed a triangle around a circle of radius R=1. What is the minimum area Avi's triangle can have?

Solution #2: The circle rotational symmetry, lead us to try out an equilateral triangle as solution If the solution is correct, stretching any side by a small amount should yield a net area increase.

Refer to the figure showing an equilateral triangle ABC with half-side of length L, circumscribed to circle of radius R=1. Stretch one of the sides by a small amount δL=AA', greatly exaggerated in the figure for clarity. From A', draw the tangent to the circle, which intersects the base of t he triangle at C'. A new triangle A'BC' is formed, with one of the sides, A'C' having a new point of tangency to the circle at N. The side AC of the equilateral triangle, tangent to the circle at M, has been stretched and rotated around the circle by an amount δθ. The stretched side A'C' is made up of two segments L_A=A'P and L_C=C'P. The stretched triangle A'BC' has gained the triangular area A'AP=ΔA_G, but lost the triangular area C'CP=ΔA_L with respect to the equilateral triangle. Expressing these areas in terms of the relevant parameters, and using the sine angle area theorem with $\sin\delta\theta \sim \delta\theta$, we have:

$\Delta A_G=(1/2)(L+R\ \delta\theta/2)\ L_A\ \delta\theta$, and $\Delta A_L=(1/2)(L-R\ \delta\theta/2)\ L_C\ \delta\theta$.

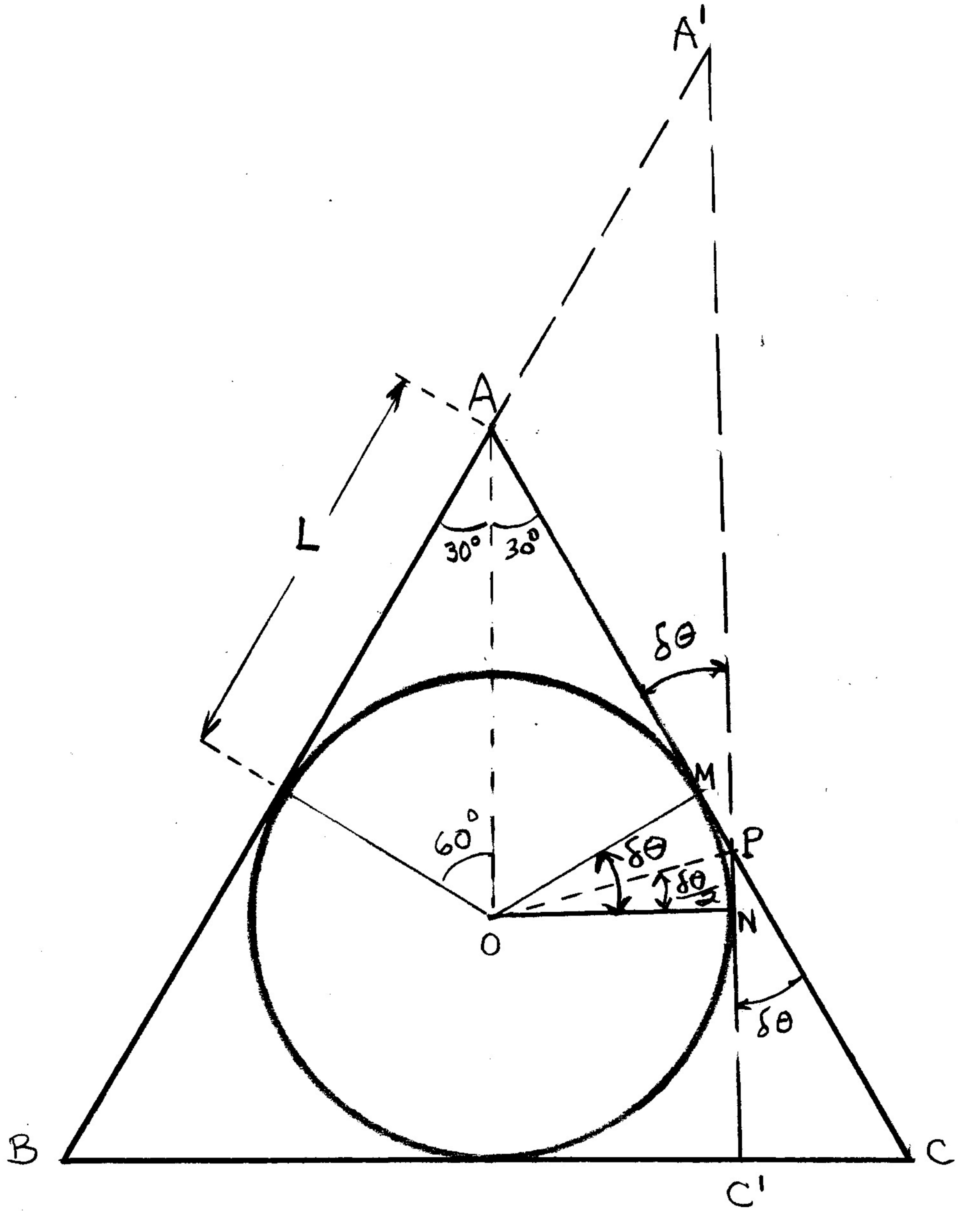
For the sought equilateral triangle to have minimum area, we must prove that $\Delta A_G>\Delta A_L$, which reduces to the following expression:

$(L_A-L_C)/(L_A+L_C)> -(R/L)\ \delta\theta/2$

Hence for the triangle to be equilateral, $L_A>L_C$. If it weren't so, and $L_A<L_C$, then $(L_C-L_A)/(L_C+L_A)<(R/L)\delta\theta/2$, but one could always find a small enough δθ to violate the inequality.
Thus, $L_A>L_C$, area $\Delta A_G>\Delta A_L$, and circumscribed equilateral triangle has minimum area. QED

Referring to the figure, the area of the sought triangle is:

A=6xArea of triangle AOM=3RL.
Using trigonometry, $L/R=\tan 60°=\sqrt{3}$, and $A=(3\sqrt{3})R^2$.
For R=1, Avi's triangle area = $3\sqrt{3}$



A'
A
L
30°
30°
δθ
M
P
60°
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δθ/2
N
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