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Subject: July 2, Puzzle Corner

Problem: Avi Ornstein has circumscribed a triangle around a circle of radius R=1. What is the minimum area Avi's triangle can have?

Solution #1: Refer to the figure which shows a scalene triangle ABC circumscribed around a circle of radius R. The center, O of the inscribed circle lies at the intersection of the triangle angle bisectors, since the segments OM, ON and OP perpendicular to the triangle sides must equal the radius, R. Call the half angles subtended by the triangle vertices A, B, and C, α, β and γ. The triangle area, S expressed in terms of the half angles is:

$S = R(\underline{AM}+\underline{BP}+\underline{CN}) = R^2(1/\tan\alpha + 1/\tan\beta + 1/\tan\gamma)$ (1a)

The normalized area $S/R^2 = y$ is the area of the triangle circumscribed around a circle of radius R = 1,

$y = (1/\tan\alpha + 1/\tan\beta + 1/\tan\gamma)$ (1b)

and is subject the constraint

$\alpha + \beta + \gamma - \pi/2 = 0.$ (2)

Form the new function Y with a Lagrange multiplier λ:

$Y = (1/\tan\alpha + 1/\tan\beta + 1/\tan\gamma) + \lambda(\alpha + \beta + \gamma - \pi/2)$ (3)

For Y to be an extremum (minimum or maximum),

$\partial Y/\partial\alpha = \partial Y/\partial\beta = \partial Y/\partial\gamma = \partial Y/\partial\lambda = 0$ (4)

which yields

$\partial Y/\partial\alpha = \lambda - 1/\sin^2\alpha = 0$ (5a)

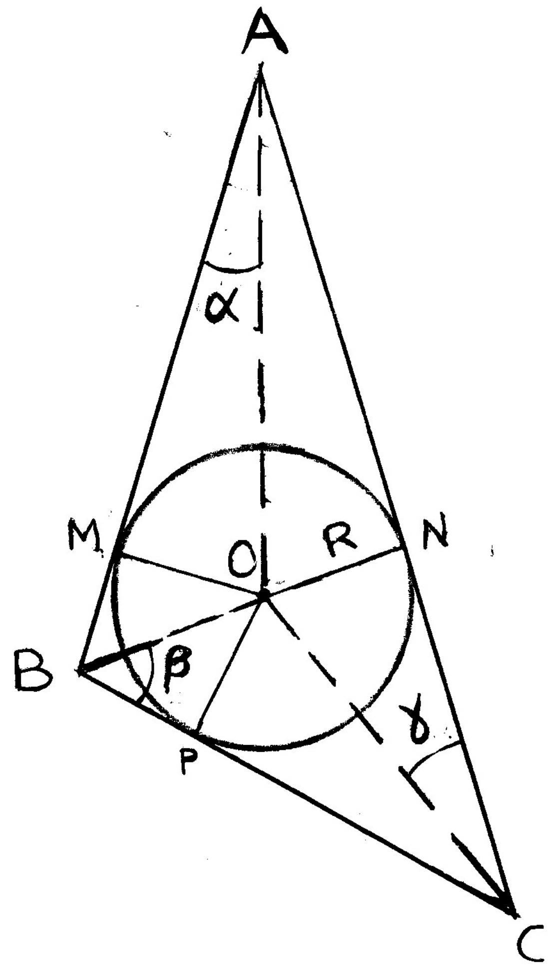
$\partial Y/\partial\beta = \lambda - 1/\sin^2\beta = 0$ (5b)

$\partial Y/\partial\gamma = \lambda - 1/\sin^2\gamma = 0$ (5c)

It follows from Eqs.5 and Eq.2 that the half angles $\alpha = \beta = \gamma = 30°$; hence the triangle is equilateral, and its normalized area is according to Eq.1b

$y = 3/\tan\alpha = 3/\tan 30° = 3\sqrt{3}$ (6)

To prove that its area is a minimum, we verify that $d^2y/d\alpha^2 > 0$. From Eq.6, $dy/d\alpha = -3/\sin^2\alpha$; and $d^2y/d\alpha^2 = 6/(\sin^2\alpha \tan\alpha) = 24\sqrt{3} > 0$ for $\alpha = 30°$. **QED**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