Puzzle Corner Dec/Jan 2, 2006

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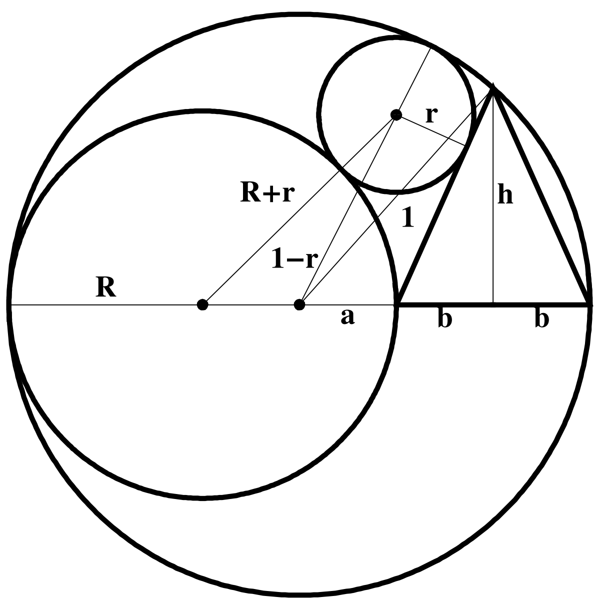
Given a circle of radius 1, place an isosceles triangle with base $2b$ and height h so that its base overlaps the right end of a diameter of the circle and its top is on the circle. Place a circle of radius R so one of its diameters coincides with the remainder of first circle's diameter. Add a third circle of radius r that is simultaneously tangent to the two circles and the interior side of the triangle. (The puzzle says: "add a third circle inscribed so that it touches the other two circles and the triangle".) The puzzle is to show the center of the third circle is directly above the interior corner of the triangle. We will actually show that there are two solutions consistent with the specification, only one of which has the desired property. Refer to the following picture:

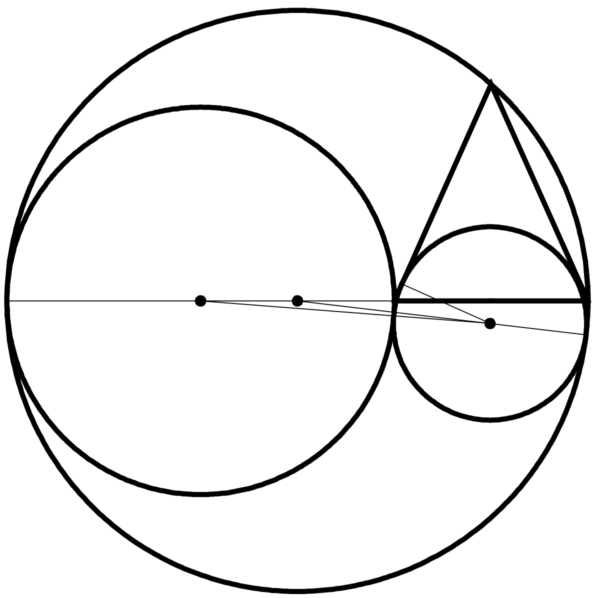


Our approach is essentially algebraic. Let (x, y) be the coordinates of the center of the third circle, relative to an origin at the interior corner of the triangle. Let a be the distance from the center of the main circle to this corner. Simple relations on the main diameter give $2R + 2b = 2$ and $a + 2b = 1$. Using that a radius is perpendicular to a tangent line: considering the line between the center of the second and third circles gives $(r+R)^2 = (x+R)^2 + y^2$. Considering the line between the center of the main and third circles gives $(1-r)^2 = (x+a)^2 + y^2$. Considering the line between the center of the main circle and the top of the triangle gives $1 = (a+b)^2 + h^2$.

The line containing the interior side of the triangle consists of all points $(0, 0) + s(b, h)$ for real numbers s. The line from the center of the third circle perpendicular to this line consists of all points $(x, y) + t(-h, b)$ for real numbers t. Equating these two gives the intersection point at $t = \frac{h\,x - b\,y}{b^2 + h^2}$. Considering the length of the segment from the center of the third circle to this point gives $r^2 = \frac{(h\,x - b\,y)^2}{b^2 + h^2}$.

Solving all these equations, leaving only b as an independent variable, gives four solutions consistent with the triangle being on the right and pointing up. One of these is: $x = 0,\ r = \frac{2\,(-1+b)\,b}{-2+b},\ y = 2\sqrt{2}\sqrt{\frac{(-1+b)^2\,b}{(-2+b)^2}} = \frac{\sqrt{2}\,r}{\sqrt{b}}$, which has the desired property that $x = 0$ and matches the picture above with $b = 1/3$.

A second solution is: $x = -\frac{4\sqrt{2}\,(-2+b)\,\sqrt{-(-2+b)^3\,(-1+b)^2\,b^4}}{b\,(-4+2\,b+b^2)^2} + \frac{4\,(-1+b)\,b\,(4-10\,b+5\,b^2)}{(-4+2\,b+b^2)^2}$, $r = \frac{4\sqrt{2}\,\sqrt{-(-2+b)^3\,(-1+b)^2\,b^4}}{(-4+2\,b+b^2)^2} + \frac{2\,(-2+b)\,(-1+b)\,b\,(4-2\,b+b^2)}{(-4+2\,b+b^2)^2}$ and $y =$ a mess. For this solution, except at the extremes, $x \neq 0$. The corresponding picture for $b = 1/3$ is:



In the two remaining solutions of the equations the third circle, located in the lower horn between the first two circles, is tangent to the side of the triangle only when the side is extended into the lower half of the main circle, which presumably violates the intent to the problem.

My statement of the problem potentially allows another type of solution where the third circle is simultaneously tangent to the first two circles at the left end of the main diameter. However, in this case the third circle is again tangent only to the extension of the triangle's side.

In all the solutions, x, r and y are constructible numbers, which implicitly gives a ruler and compass construction for the third circle.