

2002 Jul 2. The computer solution given in December lacked an important proof. James Russell closes the gap by providing an alternative, analytic solution. Consider an arbitrary player A, and divide the remaining players into two six-player sets: W, those who won against A, and L those who lost. Note that for an arbitrary player B in W, for each C in L that B lost to, (ABC) is a desired triple. Thus, we need to add up, for each B in W, the number of C in L that B lost to. We know B lost six games, and beat A, so the number of games B lost to people in L is six minus the number B lost to people in W. While we don't know that number for each individual B, we know that the six players in W played $6C2 = 15$ games against each other, so there are 15 total losses for all the B in W to other players in W. So, the sum of losses by players in W to players in L must be $6*6-15=21$, and therefore there are 21 triples containing A with the desired property. Since A was chosen arbitrarily and we identify the triples (ABC), (BCA), (CAB), the answer is $21*13/3=91$. This argument applies generally, so for $2n+1$ players, the number of triples is

$$\frac{(2n+1)}{3} \left(n^2 - \binom{n}{2} \right) = \frac{n(n+1)(2n+1)}{6} = \sum_{i=1}^n i^2$$