Puzzle corner

Send problems, solutions, and comments to Allan Gottlieb at New York University, 60 Fifth Ave., Room 316, New York, NY, 10011, or gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.

At the last minute, NYU decided to give its final exams remotely, even for those classes that (like mine) were conducted in person all semester. Apparently omicron is not giving up without a fight.

On a personal note, my wife, Alice, and I may be living part of each week in Brooklyn. My connection with that borough goes back 70 years: I grew up as a fan of the Brooklyn Dodgers. Back then my father owned a furniture store and had some baseball players as customers. I still remember going to the apartment of Willie Mays, the all-time great with the New York Giants.

Both of those teams have long since moved west to California. Time marches on.

Problems

M/A1. If you know the "Red King" bridge problem, our first offering, from Richard Thornton, will seem familiar (but not the same).

Jane, South, opens 2 no-trump and Dick, North, bids 6 no-trump. Jane sees that they have all the aces, queens, jacks, and 10s, but no kings. She can finesse East and West up to twice each and can make the contract if she can win two out of three finesses. Jane plans her play to have the highest probability of making the contract and succeeds. Before any card is played, what is the probability of her making 6 no-trump, and in what sequence did she make her finesse? Thornton offers as a hint that you do not need to know distribution probabilities to solve this problem.

M/A2. Another of Dick Hess's "logical hat problems," in which logicians each wear a hat with a number and can see every number but their own. All the logicians know that they are error-free in their reasoning. You are to figure out what numbers are on their hats.

Here, two positive odd integers are chosen that are either the same or separated by two. The larger (if any) is written on B's hat and the other on A's hat. The logicians make these statements in order.

> A1: "I don't know my number."
> B1: "I don't know my number." …
>
> A9: "I don't know my number."
> B9: "I now know my number."

M/A3 J.D. Kramer asks you to show that the angle measured clockwise from the hour hand to the minute hand in an analog clock takes on a specific value $0 < A < 360$ 11 times in every 12 hours.

Speed department

SD. Saying (a belated) goodbye to the 20th century, Sorab Vatcha asks which years in that century had the most repeated digits.

Solutions

N/D1. Daniel Glickman wants us to analyze a variant of the basketball shooting game H-O-R-S-E. Daniel and his friend Randy have equal shooting skills. That means for any given shot, they have the same probability of success. Randy shoots first, and then they alternate. The rules are:

1. Initially Randy chooses the shot they must make.
2. If Randy makes the shot and Daniel misses the (identical) shot, Randy wins.
3. If Randy makes the shot and Daniel makes the shot, Daniel wins.
4. If Randy misses the shot, then Daniel has control; he chooses the next shot and assumes the position Randy had at the beginning.

The question is: How difficult a shot should Randy choose initially? The following solution is from Jake Wildstrom.

Let us denote by q the optimal shot difficulty for Randy's chance at winning, and let p represent Randy's chance of victory with such a shot. By symmetry, when/if Daniel takes control, choosing that same shot difficulty q maximizes his own chance of winning, and so if Daniel gains control (which happens with probability $1 - q$), he will win with probability p (and thus Randy wins in that scenario with probability $1 - p$). Thus we may assert that Randy's win probability is the sum of the probability of an immediate win and a deferred win, i.e.: $p = q(1 - q) + (1 - q)(1 - p)$

This simplifies to $p = (1 - q^2)/(2 - q)$. Since q is selected to maximize p, we must find the maximum of $(1 - q^2)/(2 - q)$. A little bit of tedious calculus later, we find that a shot with probability $2 - \sqrt{3}$ of success (about 0.268) is the value of q in the range $[0, 1]$ which corresponds to a local maximum, and that its associated maximum likelihood of winning $p = 4 - 2\sqrt{3}$, or approximately 54%, exceeds both the 0% chance of victory at $q = 1$ and the (somewhat notional) 50% chance of victory at $q = 0$.

N/D2. Our second problem was from Pericles Manglis, who wanted you to find the nth power and the nth root of the simple square matrix

$$\begin{matrix} (A + B)/2 & (A - B)/2 \\ (A - B)/2 & (A + B)/2 \end{matrix}$$

Several readers (and your editor) found this result surprising, but that did not prevent Burgess Rhodes from solving it. Decompose M in the following way:

$$M(A, B) = A\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} + B\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} = AP + BR$$

Matrices P and R satisfy

$$P^2 = P, \ R^2 = R, \text{ and } PR = RP = 0$$

For any number $x, y, z,$

$$\begin{aligned} M(w, x) \times M(y, z) &= [wP = xR] \times [yP + zR] \\ &= wyP^2 + wzPR + xyRP + xzR^2 \\ &= wyP + xzR \\ &= M(wy, xz) \end{aligned}$$

Thus

$$M^2(A, B) = M(A^2, B^2)$$

and in general

$$M^n(A, B) = M(A^n, B^n).$$

It follows that the nth root of M(A, B) is M($A^{1/n}$, $B^{1/n}$) because

$$M^n(A^{1/n}, B^{1/n}) = M((A^{1/n})^n, (B^{1/n})^n) = M(A, B).$$

N/D3. Here you were to find the longest ladder that can be taken around a two-dimensional 90° corner between two halls of width a and b.

The following solution from Anthony Bielecki is elegant and appears straightforward—assuming you remember all the trigonometry you once learned.
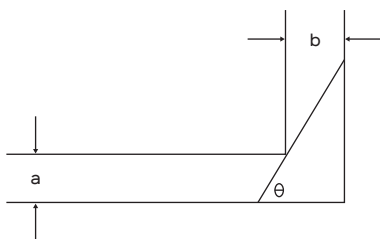
With θ as shown in the diagram, the length from outer wall to outer wall is

$$L = \frac{a}{\sin\theta} + \frac{B}{\cos\theta}$$

The ladder must be able to pass the shortest L, found by setting the derivative to zero:

$$\frac{dL}{d\theta} = -a\frac{\cos\theta}{\sin^2\theta} + b\frac{\sin\theta}{\cos^2\theta} = 0$$

Solve for θ

$$\frac{a}{b} = \frac{\sin^3\theta}{\cos^3\theta} = \tan^3\theta$$

$$0 = \arctan\left(\left(\frac{a}{b}\right)^{1/3}\right)$$

Substitute into the first equation

$$L = \frac{a}{\sin\arctan\left(\left(\frac{a}{b}\right)^{1/3}\right)} + \frac{b}{\cos\left(\arctan\left(\left(\frac{a}{b}\right)^{1/3}\right)\right)}$$

Use the trigonomic identities

$$\sin(\arctan(x)) = x \,(1+x^2)^{-1/2} \ \text{ and } \ \cos(\arctan(x)) = (1+x^2)^{-1/2}$$

to obtain

$$L = \frac{a}{\left(\frac{a}{b}\right)^{1/3}\left(1+\left(\frac{a}{b}\right)^{2/3}\right)^{-1/2}} + \frac{b}{\left(1+\left(\frac{a}{b}\right)^{2/3}\right)^{-1/2}}$$

Since a and b are interchangeable, we expect that L can be expressed in a symmetric form. Indeed, with simple algebra, the above expression can be manipulated to obtain:

$$L = (a^{2/3} + b^{2/3})^{3/2}$$

For the trivial case of $a = b$, this yields the expected $L = a \cdot 2\sqrt{2}$ (and θ = 45°).

Other responders

R. Akkurd, P. Alto, M. Barr, S. Berkenblit, L. Bonderson, M. Brill, D. Cane, P. Cassady, J. Chandler, M. Chartier, T. Chow, F. Cogswell, G. Coram, J. d'Almeida, R. Deming, N. Derby, M. Douma, E. Field, G. Fischer D. Fitterman, J. Freilich, M. Gordy, T. Griffin, J. Grossman, T. Hafer, J. Hardis, J. Harmse, P. Hartmann, S. Helm, W. Hill, A. Hirshberg, H. Hodara, R. Keller, J. Knox, E. Koch, P. Kramer, A. Kunin, J. Larsen, A. Levin, R. Lipes, J. Macro, P. Manglis, F. Marcoline, T. Mattick, B. Mellinger, Z. Mester, T. Mita, G. Muldowney. R. Orr, C. Ouellette, J. Reed, K. Rosato, R. Rosen, J. Rulnick, F. Schmitt, S. Shalom, R. Shapiro, S. Siberberg, E. Signorelli, B. Snyder, S. Sperry, W. Stein, A. Stern, U. Sukhatme, U. Team, C. Wampler, M. Weiss, T. Woods, J. Wrinn.

Solution to speed problem

1911 and 1999, each with three repeated digits.