Puzzle corner

Send problems, solutions, and comments to Allan Gottlieb at New York University, 60 Fifth Ave., Room 316, New York, NY, 10011, or gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.

This being the first issue of a calendar year, we again offer a "yearly problem" in which you are to express small integers in terms of the digits of the new year (2, 0, 2, and 2) and the arithmetic operators. The problem is formally stated in the "Problems" section, and the solution to the 2021 yearly problem is in the "Solutions" section.

Problems

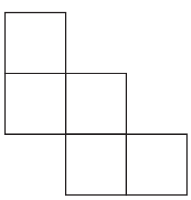
Y2022. How many integers from 1 to 100 can you form using the digits 2, 0, 2, and 2 exactly once each, along with the operators +, −, × (multiplication), ÷ (division), and exponentiation? We desire solutions containing the minimum number of operators; among solutions having a given number of operators, those using the digits in the order 2, 0, 2, 2 are preferred.

Parentheses may be used; they do not count as operators. A leading minus sign, however, does count as an operator.

J/F1. Andy Schwartz must have taken the math sequence in a different order than I did. To help his son calculate the integral of $\sqrt{1 + x^2}$, he left the beaten path and tried the substitution $x = i\sin(u)$. His challenge for us is to find a complex analytic expression for $\arcsin(x)$.

J/F2. Our last regular problem is another "modest polyomino" from Richard Hess and Robert Wainwright.

You are to design a connected tile so that three of them cover the shape below. The tiles are identical in size and shape and may be turned over so that some are mirror images of the others. They must not overlap each other or the border of the polyomino.



Speed department

SD. Ram Gopalan seeks two irrational numbers, x and y, such that x^y is rational.

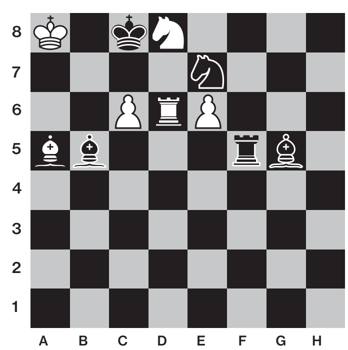
Solutions

Y2021. The following list comes from Michael Geradi.

$1 = 2^{0\times 21}$	$11 = 20/2 + 1$	$32 = 20 + 12$
$2 = 2 \times 0 + 2 \times 1$	$13 = 12 + 2^0$	$39 = 2 \times 20 - 1$
$3 = 2^0 + 2 \times 1$	$14 = 2^2 + 10$	$40 = 20 \times 2^1$
$4 = 2^{0+2} \times 1$	$18 = 20 - 2 \times 1$	$41 = 20 + 21$
$5 = 2^{0+2} + 1$	$19 = 21 - 2 + 0$	$42 = 2 \times 21 + 0$
$6 = 10 - 2^2$	$20 = 20(2 - 1)$	$50 = 10^2/2$
$7 = 10/2 + 2$	$21 = 20 + 2 - 1$	$51 = 102/2$
$8 = 2^{0+2+1}$	$22 = 22 + 1 \times 0$	$60 = 120/2$
$9 = 20/2 - 1$	$23 = 22 + 1^0$	$98 = 10^2 - 2$
$10 = 10 - 2 + 2$	$24 = 2 \times 12 + 0$	$100 = 102 - 2$

S/O1. Apparently some chess players consider the ordinary rules dull and offer variations for us to try. One variant, sometimes called "free-capture chess," permits players to capture their own pieces and pawns as well as the opponent's. However, players cannot check or capture their own king. Consider the following position in free-capture chess, with White to move.

How does White force a mate in two?



Jim Larsen found the desired mate in two while noting the existence of a mate in one. He writes:

There is no first move by White that can force the Black king to move, so mate must occur with BK remaining at c8. Only the WN and WR can be moved to check on move 2, but all the potential WN checking possibilities can be thwarted by Black on its next move, leaving WR as the only possible checking piece.

The obvious move 1 is WR to d8 mate, but this is mate in one, not two. Moving any other piece except WR as a delaying move allows BR to d5 to defend the ultimate checking move of WR to d8. Move 1 must therefore be WR to d7. Black can defend a move 2 checking move of WR x WN with BR to d5 or a move 2 checking move of WR to c7 with BB to f4, but it cannot defend against both.

Mate will occur on move 2 with either WR x WN or WR to c7, depending on Black's move.

S/O2. Richard Lipes sent us the following number problem. Alice lives on a long street in which all the houses are on one side, and the houses are numbered consecutively (i.e., 1, 2, ... *N*). Alice, a numbers whiz, noticed that the sum of all the house numbers below her house equals the sum of all the house numbers above her house. Her house number has three digits. What is it?

John Reed, with "a little help from my friend" (in this case Mathematica), solved the problem quite easily.

I use the formula for the sum of integers from 1 to *n*:

$$S = \frac{n(n+1)}{2}$$

For the addresses up to Alice's, the sum is:

$$S1 = \frac{A(A-1)}{2}$$

Where *A* is Alice's address.

For addresses greater than Alice's up to the last address, *N*, the sum is:

$$S2 = \frac{N(N+1)}{2} - \frac{A(A+1)}{2}$$

Setting $S1 = S2$ and solving for *A*, this equation simplifies to:

$$A = \sqrt{\frac{N(N+1)}{2}}$$

The problem now is to find the value of *N* that gives an integer value when the square root is taken. I used Mathematica to find this. It found a value of 288. This gives a value of Alice's address of 204.

S/O3. The following two-part cryptarithmetic problem is from David Singmaster's book *Problems for Metagrobologists*. Can two ODDs make an EVEN in the cryptarithmetic sense? That is, try to substitute digits for letters and solve

ODD + ODD = EVEN

Part two is to arrange for three ODDs to make an ODD cryptarithmetically.

Several authors noted that part two is not precise, and two obvious solutions are

ODD + ODD ODD = ODD and ODD ODD/ODD = ODD.

Aaron Hirshberg sent us the following solution to part one:

ODD + ODD = EVEN

D has to be >= 5. If D < 5, then D + D = N in the ones column, and D + D = E in tens column.

Assume D = 5. In the ones column, D + D % 10 = 0. Therefore N = 0. The carry is 1. (D + D + 1) % 10 in the tens column = 1. Therefore E = 1. This works really well, because the thousands column in the answer better be a 1 or there won't be an answer. The carry (D + D + 1) / 10 over to the hundreds column is 1. So (O + O + 1) % 10 in the hundreds column is V, and (O + O + 1) / 10 in the thousands column is 1. If O is 8, V is 7, and we have 855 + 855 = 1,710. If O is 6 and V is 3, we have 655 + 655 = 1,310.

Assume D = 6. In the ones column, D + D % 10 = 2. Therefore N = 2. The carry is 1. (D + D + 1) % 10 in the tens column is 3. E cannot be 3. O + O + the carry from the hundreds column cannot be 3. This also applies to D = 7, 8, or 9. E will be too large.

Better late than never

J/A SD Peter Blicher points out that the sun sets at various angles depending on latitude and time of year, so the question does not have enough information for a solution.

Other responders

F. Albisu, A. Alcala, B. Andeen, R. Anderson, M. Barr, T. Barrows, S. Bekenblit, E. Berman. K. Bernstein, L. Bonderson, M. Brill, E. Browne, W. Brumi, R. Byrne, D. Cane, J. Casalegno, J. Chandler, F. Cogswell, D. Craig, D. Dewan, M. Douma, S. Fetter, M. Galatin, L. Griffith, J. Grossman, J. Hardis, T. Harriman, A. Hirshberg, J. Horton, E. Kaplan, P. Karmer, J. Knox, J. Labuz, N. Lang, J. Larsen, Z. Levine, B. Lichtenstein, D. Linden, J. Mackro, T. Maloney, P. Manglis, D. Mellinger, T. Mita, D. Morgen, A. Moulton, G. Muldowney, J. Ng, F. Olenberger, A. Ornstein, R. Orr, C. Ouellette, R. Park, Prauch, C. Peski, J. Raines, I. Resnikoff, B. Rhodes, L. Schaider, S. Silberberg, B. Snyder, S. Spitz, A. Stern, W. Sun, S. Swaminathan, D. Turek, S. Ulens, S. Vatcha, R. Whitman, R. Wilkins, J. Winters, D. Worley, K. Zeger.

Solution to speed problem

Let $x = \sqrt{2}$

If x^x is rational, we are done. If it is irrational, then $(x^x)^x = 2$ solves the problem.