Puzzle corner

Send problems, solutions, and comments to Allan Gottlieb at New York University, 60 Fifth Ave., Room 316, New York, NY, 10011, or gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.

Stanley Liu proposed the following as a speed problem, but I found the result very surprising and want to share it here. For more information see //oeis.org/A0020024.

Consider the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ... where the number n appears n times. It is fairly clear that the nth number ends at about the n^2 term, but I was surprised to see that term k is *exactly* $\lfloor \sqrt{2k} + \frac{1}{2} \rfloor$.

Problems

J/A1. Duffy O'Craven offers what I might call our first "tit for tat" chess problem. A few years ago O'Craven submitted our only "helplessmate" problem.

This time you are to find a legal chess position where the player-to-move is in check and this player's only legal move is to deliver checkmate. The mating move must be neither a capture nor a discovered check.

J/A2. Our next problem is a cryptarithmetic offering from David Dewan. I will give preference to solutions that are not computer searches of all the possibilities.

In the equation

$$HA^{PPY} = NEW + YEAR$$

you are to substitute a digit 0 to 9 for each letter. Distinct letters get distinct digits, and if a letter appears multiple times the same digit is substituted each time.

J/A3. We close with a geometry problem from Burgess Rhodes.

All lines intersecting at a fixed point in R^3 fill R^3. No two of these lines are parallel. Also, all lines parallel to a fixed line in R^3 fill R^3. No two of these lines intersect.

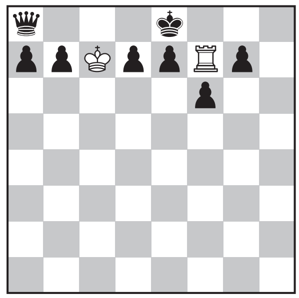
The question is whether R^3 can be filled with lines no two of which are parallel *and* no two of which intersect.

Speed department

SD. Ermanno Signorelli wonders why computer scientists confuse Halloween and Christmas.

Solutions

Player 2



Player 1

M/A1. This "Turnary Reasoning" problem asks if the above position can be reached in a legal chess game and, if so, whose move it is.

Rich Downey shows that the position cannot be legally reached by showing that no "last move" could have resulted in the given position. Specifically, Downey asks separately if each piece could have made the last move.

—The White king: **No**. For any of the six squares the White king could have moved from, it would have been in check. However, there is no previous Black move that would have put the White king into check. Thus the White king would have been in check after White's last move.

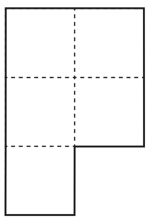
—The White rook: **No.** There is only one square the White rook could have moved from. However, if it had been there, the Black king would have been in check after Black's previous move.

—The Black pawns: **No**. Five of them have never moved, and the sixth could not have moved from the square where the White rook is.

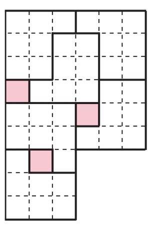
—The Black queen: **No**. Any of the three squares it could have moved from would have had White's king in check after White's last move.

—The Black king: **No.** It could not have moved from the square adjacent to the White rook, since the rook would have been there first, and hence the Black king would have been in check after Black's previous move. It could not have moved from the square adjacent to the White king, since kings can never be adjacent.

M/A2. In this "modest polyomino" problem from Richard Hess and Robert Wainwright, you are required to design a connected tile so that six of them cover at least 93% of the area of the pentomino given below. The tiles are identical in size and shape and may be turned over so that some are mirror images of the others. They must not overlap each other or the border of the pentomino.



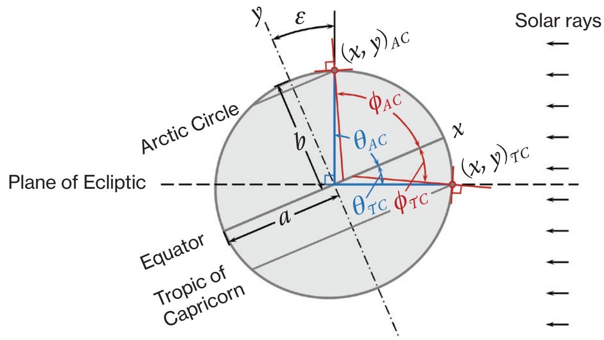
Joe Catling's solution (below) begins by dividing each of the five squares in the original figure into nine sub-squares, giving 45 subsquares in total. He then defines a seven-sub-square tile and places six of them as shown. These six tiles cover 7 × 6 = 42 sub-squares, which have a total area equal to 42/45 > 93% of the area of the original figure. The three sub-squares not covered are highlighted in the diagram.



M/A3. Lee Giesecke presents a venture into astronomy and 3D geometry. Imagine it's the winter solstice and you want to compute the geocentric and geodetic latitudes of the Arctic Circle and the Tropic of Capricorn. The geocentric latitude gives the angle between the equatorial plane and a line from Earth's center to a point on the surface. The geodetic latitude assumes a line from the same surface point that is perpendicular to a plane tangent to Earth's surface. The angle of intersection of the line with the equatorial plane gives the geodetic latitude. Assume Earth can be represented by an ellipsoid of revolution with the semi-major and semi-minor axes $a = 1$ and $b = 0.99665$, and that Earth's obliquity (axial tilt with respect to the ecliptic) is 23.44°.

When I selected this problem, I felt it was quite interesting, but also challenging. I was pleasantly surprised that I received three carefully prepared solutions. Well done, readers! Greg Muldowney's solution is below; those from Joe Catling and Burgess Rhodes are on the "Puzzle Corner" website at cs.nyu.edu/~gottlieb/tr.

As described, the Earth is slightly larger across the equator than between the poles. In terms of x-y coordinates fixed at the center and the semi-axes a and b, the surface of the Earth is the ellipse $(x/a)^2 + (y/b)^2 = 1$. The ellipse is also tilted by $\varepsilon = 23.44°$ versus the plane of orbit around the sun.



At the winter solstice, the Arctic Circle corresponds to the greatest latitude above the equator at which the sun is visible, while the Tropic of Capricorn is the greatest latitude below the equator at which the sun appears directly overhead. Thus the geocentric latitudes are $\theta_{AC} = (90 - \varepsilon)$ or 67.56° north, and $\theta_{TC} = \varepsilon$ or 23.44° south. These two latitudes have associated surface points $(x, y)_{AC}$ and $(x, y)_{TC}$ respectively, at which $(y/x) = \tan\theta$.

For geodetic latitude, the slope of the surface tangent at (x, y) is found by differentiating the ellipse equation implicitly to give $2x/a^2 + 2y(dy/dx)/b^2 = 0$, thus $(dy/dx) = -b^2x/a^2y = -(b/a)^2 \cot\theta$. A line perpendicular to the surface at (x, y) therefore has a slope of $[-(dy/dx)^{-1}] = (a/b)^2 \tan\theta$. This slope is exactly the tangent of the geodetic latitude ϕ, hence $\phi = \tan^{-1}[(a/b)^2 \tan\theta]$. Applying this relationship at $(x, y)_{AC}$ and $(x, y)_{TC}$ gives:

$$\phi_{AC} = \tan^{-1}\left[\left(\frac{a}{b}\right)^2 \tan\left(\frac{\pi}{2}\right) - \varepsilon\right] = 66.70° \text{ N}$$

$$\phi_{TC} = \tan^{-1}\left[\left(\frac{a}{b}\right)^2 \tan(\varepsilon)\right] = 23.58° \text{ S}$$

The geodetic latitudes slightly exceed the geocentric latitudes, consistent with Earth's greater equatorial versus polar dimension.

Other responders

J. Brown, T. Keske, P. Kramer, J. Larsen, Z. Levine, T. Mita, and A. Ornstein.

Solution to speed problem

Because computer science types know that 31 Oct = 25 Dec (i.e., 31 base 8 = 25 base 10). Recall that speed problems are "often whimsical."