Puzzle corner

Send problems, solutions, and comments to Allan Gottlieb at New York University, 60 Fifth Ave., Room 316, New York, NY, 10011, or gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.

As I finalize this column in early March, I hope that when you read it in May/June, the worldwide incidence of Covid-19 will be steadily decreasing. Intellectually, I realize that infectious diseases can spread rapidly, but it is jolting to see it happen. I extend my best wishes and sympathies to everyone affected.

Problems

M/J1. We begin with a bridge problem from Larry Kells. Assume you hold:

Spades:	AQ10
Hearts:	AKQ
Diamonds:	AKQ
Clubs:	AKQJ

You bid 6 no-trump and play it there.

Can you make this contract against any distribution of cards to the remaining three hands assuming best play on all sides? You should also assume you and the opponents know the distribution.

M/J2. Our second offering is an unusual pentominoes problem from H. Yamamoto (via that most prolific puzzler Nob Yoshigahara). First choose one of the 12 pentominoes as your tile. Recall that a pentomino contains five 1×1 squares and consider an 8×8 board, initially empty. What is the maximum number of your tiles that can be placed on the board without overlap? The answer depends on your choice of tile, so a full solution consists of determining the maximum for each of the 12 distinct pentominoes.

M/J3. Bruce Heflinger has a question about the terms in the Fibonacci sequence. This well-known sequence begins 1, 1, 2, 3, 5 ... and is defined by the equations $F(0) = F(1) = 1$ and for all $n >= 2$, $F(n) = F(n-1) + F(n-2)$. Heflinger asks you to show that any two adjacent numbers in the sequence are relatively prime (i.e., they share no common factor other than 1).

Speed department

Sorab R. Vatcha wants you to find two different sets each containing three unequal integers such that, for each set, the three numbers have the same sum and product.

Solutions

J/F1. Larry Kells wants you to construct a single full deal (i.e., specify all four hands) where, with South as declarer, the opponents can defeat every possible contract—and to maximize the number of high-card points South can hold in such a deal. To be clear, with this one full deal any contract by South can be defeated with best play on both sides.

I report two solutions. The second looks "too good to be true," but I am not sufficiently knowledgeable about bridge to be sure.

This first solution, from Jim Larsen, gives a 22-point hand.

In the following hand, South has 22 points, the maximum high-card points he/she can have with the condition that the opponents can defeat any possible contract:

North
♠ 7 6 5
♥ 6 5 4
♦ 4 3 2
♣ 5 4 3 2

West
♠ Q J 10 9 8
♥ Q J 10 9
♦ K Q
♣ K Q

East
♠
♥ 8 7
♦ 10 9 8 7 6 5
♣ 10 9 8 7 6

South
♠ A K 4 3 2
♥ A K 3 2
♦ A J
♣ A J

South can win six tricks, but no more. No-trump play is straightforward, with South getting six top tricks and West getting the remainder. Spades or hearts as trump play essentially the same way. Clubs are the most advantageous other trump choice, but when North eventually gets a chance to ruff, East can over-ruff, pull North's remaining trump, and either run diamonds or return the hand to West-South control. There is no arrangement where a Q can be substituted for a J without South getting an extra trick.

Our second solution, from Bob Wake, is a 30-point dream hand.

North
♠ 4 3 2
♥ 3 2
♦ 6 5 4 3
♣ 6 5 4 3

West
♠ 10 9 8 7 6 5
♥
♦ 2
♣ A Q 10 9 8 7

East
♠
♥ 10 9 8 7 6 5 4
♦ A 10 9 8 7
♣ 2

South
♠ A K Q J
♥ A K Q J
♦ K Q J
♣ K J

East-West can take eight tricks in clubs, and seven in every other suit. Against a spade, heart, or no-trump contract, West can lead a diamond to the ace, East returns a club, and West plays high clubs at every opportunity. South is forced to ruff twice if spades are trump, or once if hearts are trump, setting up the needed four trump tricks either way. Against a diamond contract, West leads a spade, East ruffs and returns a heart for West to ruff, West leads another spade, East ruffs and leads a club, and then West takes two clubs and forces South to ruff, giving East the fourth trump trick they need. Finally, to get that eighth trick in clubs, West leads a spade for East to ruff, East then leads a heart for West to ruff, diamond to the ace at trick three; West ruffs a red card at trick four and exits with a spade. South has too many red cards left to avoid the endplay if West ruffs and exits with a spade at every opportunity. (East-West also have long-shot chances to take extra tricks against a red-suit contract against subpar play by drawing trumps, but if West leads diamonds against a diamond contract, East ducks, and South realizes the need to respond by breaking clubs and ends up making the contract.)

J/F2. Richard Thornton sometimes overpays, since he occasionally multiplies the costs of individual items instead of summing them. (We assume all items cost a positive-integer multiple of cents.)

One time, he purchased four items whose total cost is \$7.11, but he was lucky since the product was also \$7.11. What did the individual items cost?

Thornton also asks a more challenging question. There are many examples of four item costs (again, each a positive-integer number of cents) with the sum equal to the product. Which of these sets of four costs gives the largest sum? Which gives the smallest?

Richard Lipes sent us the following unique solution approach he received from a friend, who worked on the problem and then replied to Lipes: "After some trial and error, I used the new [his name] approach, which works like this. Go to Google, enter 'Advanced Search,' and then enter in the first field 'Diophantine \$7.11.' Works like a charm!"

Indeed it does! But I find the more traditional method, "figure it out yourself," to be more satisfying, although admittedly more time consuming. As Lipes mentions, the Google search does reveal that significant work has been done on this problem.

The following satisfying solution is from Greg Muldowney.

If four items cost *a, b, c,* and *d* cents respectively, in dollars and cents the sum is $(a + b + c + d)/100$, whereas the product is $abcd/(100)^4$. For these to equate, the individual costs must satisfy $(a + b + c + d) = abcd/(100)^3 = abcd/(5^6 \times 2^6)$. In Thornton's case both sides are 711. Therefore *abcd* comprises the factors of 711—that is, 79×3^2, as well as $5^6 \times 2^6$. These 15 integers are to be parsed into sub-products *a*, *b*, *c*, and *d* that sum to 711. Not all can have 5 as a factor—one must end in 1 or 6. Having factors of 5 in three costs, notably as 5^3, 5^2, and 5^1, leads to the solution:

$$\begin{aligned} abcd &= (79 \times 3^2)(5^6 \times 2^6) \\ &= (5^3)(5^2 \times 3 \times 2)(5 \times 3 \times 2^3)(79 \times 2^2) \\ &= (125)(150)(120)(316) \\ a + b + c + d &= 125 + 150 + 120 + 316 = 711 \end{aligned}$$

Thornton's item costs were then \$1.20, \$1.25, \$1.50, and \$3.16.

The largest sum of four item costs is deduced by solving the sum-product equality, $(a + b + c + d) = abcd\ (100)^3$, for the one dependent cost (say *d*) and using it to express the sum (in cents):

$$d = \frac{a+b+c}{\left(\frac{abc}{10^6}\right) - 1} \longrightarrow S = \frac{(a+b+c)\ abc}{abc - 10^6}$$

Maximal *S* values are implied at $abc = 10^6 + 1$. Further, for fixed *abc*, the term $(a + b + c)$ is greatest if $a = b = 1$. The absolute largest sum *S* therefore has $(a, b, c) = (1, 1, 1000001)$, and:

$$d = \frac{1 + 1 + 1{,}000{,}001}{\left(\frac{1{,}000{,}001}{10^6}\right) - 1} = \frac{1{,}000{,}003}{10^6} = 1{,}000{,}003{,}000{,}000$$

$$S = \frac{(1 + 1 + 1{,}000{,}001) \times 1{,}000{,}001}{1{,}000{,}001 - 10^6} = 1{,}000{,}004{,}000{,}003$$

Thus \$10,000,040,000.03 is both the sum and the product of the item costs \$0.01, \$0.01, \$10,000.01, and \$10,000,030,000.00.

The smallest sum of four costs that matches the product occurs when all costs are equal. In this case $4d = d^4/(100)^3$, from which $d = 100 \times 4^{1/3} = 158.74$ and S = 634.96. Therefore each integer from 635 upward (except primes such as 641 and 643) is factored into all possible sub-products along with $2^6 \times 5^6$, and combinations of these tested for sum-product equality. The first feasible case is found at $644 = 2^2 \times 7 \times 23$, or \$6.44, with item costs of \$1.25, \$1.75, \$1.60, and \$1.84—multiples of 5^3, 5^2, 5^1, and 5^0, respectively.

Better late than never

Y2019. John Chandler sent the following improvements.

$$\begin{aligned} 8 &= 9 - 12^0 \\ 9 &= 21^0 \times 9 \\ 14 &= 10/2 + 9 \\ 18 &= 19 - 2^0 \\ 19 &= 29 - 10 \\ 93 &= 102 - 9 \end{aligned}$$

N/D3. Jim Williams recommends the following video solution: https://www.youtube.com/watch?v=HQc-54hQ8kw.

Other responders

S. Alexander, R. Anderson, M. Branicky, B. Chapp, N. Derby, D. Forouhari, H. Gries, T. Hafer, T. Harriman, D. Mellinger, T. Mita, B. Rhodes, L. Schaider, E. Signorelli, S. Sperry, and L. Tatevossian.

Solution to speed problem

{1, 2, 3} and {−1, −2, −3}