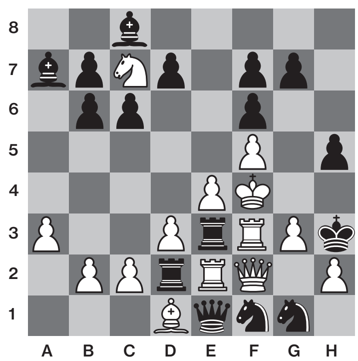
Puzzle corner

Send problems, solutions, and comments to Allan Gottlieb at New York University, 60 Fifth Ave., Room 316, New York, NY, 10011, or gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.

Oops. I must apologize for misprinting the chess diagram for M/A 1. The four pieces on the first rank were each erroneously shifted right one square. The correct problem is offered just below.

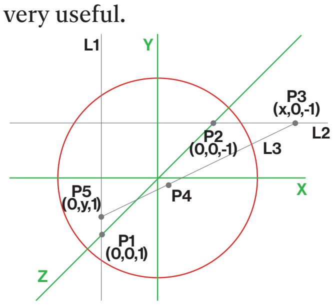
Problems

J/A 1. Another "Whose turn is it?" chess problem from Timothy Chow. Can the diagrammed position be reached by a sequence of legal moves from the standard starting position in chess? If so, can you determine whose turn it is to move?



J/A 2. Stephen R. Shalom has a large family and strange (to me) greeting customs. He reports that 18 members of his extended family (none of whose age is over 100) came to visit. They arrived two at a time. Their ages consisted of 18 different integers. As each pair arrived he computed the difference of the squares of their ages and, interestingly, got the same result every time. What were the 18 ages?

J/A 3. Robert Bird offers a problem for which a good 3D geometric sense is very useful.



A sphere of radius 1 has its center at the origin of an xyz system.

Line L_1 lies in the yz plane, is parallel to the y axis, and is tangent to the sphere at $P_1 = (0, 0, 1)$.

Line L_2 lies in the xz plane, is parallel to the x axis, and is tangent to the sphere at $P_2 = (0, 0, -1)$.

Line L_3 leaves L_2 at $P_3 = (x, 0, -1)$, is tangent to the sphere at P_4, and intersects L_1 at $P_5 = (0, y, 1)$.

You are to find y in terms of x.

Speed department

John Astolfi wishes to drive a "tour" of the continental United States, starting and ending in the same state and visiting each of the other 47 only once. How should he proceed?

Solutions

M/A 1. As mentioned, I misdrew the diagram into a clearly unreachable state. The problem has been reopened as J/A 1.

M/A 2. Nob Yashigahara wants you to place the digits 1 through 9 once each into the nine boxes to yield a valid equation.

$$\frac{\square}{\square\,\square} + \frac{\square}{\square\,\square} + \frac{\square}{\square\,\square} = 1$$

It seemed at first that personal computers are too powerful for these problems. A first cut says 10 digits to place in nine slots yields "only" 10^9—i.e., a billion—possibilities. We can try them all.

A second cut notices that 0 is not allowed, nor are duplicates, so there are only 9! = 362,880 possibilities.

However, I did receive some rather more analytical solutions; the most extreme, from David Emmes, is on the Puzzle Corner website. There is also the search-based solution from Burgess H. Rhodes, who notes that the two boxes in the denominator can have different interpretations. What follows from Kenneth Rios is a solution done essentially by hand. Its only disadvantage is that his uniqueness claim is just "highly likely."

"I began by selecting the largest possible fraction that can be constructed using three of the digits, which is 9/12. The problem then reduces to placing the digits 3, 4, 5, 6, 7, 8 into the six boxes below such that

$$\frac{\square}{\square\square} + \frac{\square}{\square\square} = \frac{1}{4}$$

"From the remaining digits I picked the two smallest candidate denominators available (without digit replacement), 34 and 56, but 56 poses a problem since 7/34 and 8/34 are far too big for their corresponding fractions, 8/56 and 7/56. For that reason I also rejected 57 and 58 as denominators, conditional on 34 being a denominator.

"While still conditioning on 34, I proceeded to the next smallest candidate denominators: 65, 67, and 68. Since 68 is a multiple of 34, I tried both 7/34 + 5/68 and 5/34 + 7/68. In the latter case, it is easy to verify that 5/34 + 7/68 = 17/68 = 1/4.

"Thus, the final answer is 9/12 + 5/34 + 7/68 = 1.

"Even without further analysis or simulation, it is highly likely to me that this is the only solution (the fractions themselves can be reordered, of course)."

M/A 3. In a number of two-person games, such as tennis or table tennis, a game can reach a "deuce" state in which the score is tied and the game proceeds as follows. Players alternate serving, and each serve leads to a point for one of the players. When one is up by two points, that player wins the game.

Assume player A, when serving, wins the point with probability p_A and player B, when serving, wins the point with probability p_B.

What is the probability that player A wins the game, assuming that A serves first at deuce? Is it better to serve first at deuce or second?

Here's Benny Cheng's solution. Let us assume that player A wins a point with probability p_A when serving, and similarly, assume player B wins a point with probability p_B at serve. Then player A wins with probability P(A wins) = $p_A(1 - p_B)/(p_A + p_B - 2p_A p_B)$.

Furthermore, the above probability also holds when player A is receiving service, so it does not matter who serves first at deuce.

To show this, we apply Markov chain methods to arrive at the solution. Suppose player A is serving, and let state 1 be the deuce state. State 2 is the state where player A is ahead by one point. State 3 is the state where player B is ahead by one point. State 4 is the state where player A is ahead by 2 points and hence wins the game. Finally, state 5 is the state where player B wins the game. Note that states 4 and 5 are absorbing states. The state transition matrix for this Markov chain looks like the following:

$$\pi = \begin{pmatrix} 0 & p_A & 1-p_A & 0 & 0 \\ p_B & 0 & 0 & 1-p_B & 0 \\ 1-p_B & 0 & 0 & 0 & p_B \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We are interested in states 4 and 5, which can only happen with an even number of transitions; hence it suffices to consider the steady-state solution

$$x_S = \lim_{n\to\infty} x\,\pi^{2n},$$

where x=(1,0,0,0,0) is the initial-state vector at deuce. Straightforward calculations show that

$$(x_s)_4 = \frac{p_A(1-p_B)}{p_A + p_B - 2p_A p_B} = \text{P(A wins at serve)}$$

as claimed. It is also easily verified that if instead player B is the server at deuce, then the chance of player A winning is

P(A wins receiving service) = P(B loses at serve)

= 1−P(B wins at serve)

$$= 1 - \frac{p_B(1-p_A)}{p_A + p_B - 2p_A p_B} = \frac{p_A(1-p_B)}{p_A + p_B - 2p_A p_B}$$

= P(A wins at serve)

Better late than never

2017 S/O 1. Tom Terwilliger believes that only 10, not 11, tricks are possible.

2018 S/O 1. Bart Bramley believes that 7 no-trump can be made with the given layout.

2019 M/A SD. Mark Henneberger suggests that the woman was providing her age in base 9. Walt Bilofsky notes that using the International Date Line is an alternative.

Other responders

Responses have also been received from M. Branicky, C. Chambers, J. Chandler, T. Chase, G. Coss, J. DiBella, D. Fitterman, H. Garber, I. Gershkoff, M. Gordy, J. Grossman, A. Guthmiller, K. Haruta, A. Hirschberg, P. Howard, S. Laxton, M. Lewandowski, J. Light, M. Marcus, M. Marinan, J. Martinez, D. Mellinger, Z. Mester, I. Ming, T. Mita, R. Morgan, A. Ornstein, A. Ritter, A. Sezginer, A. Shuchat, E. Signorelli, T. Sim, W. Stein, A. Stern, and L. Wagner.

Solution to speed problem

He should give up. New York blocks the New England states from the rest of the country. Once you leave New England, you can't get back in; once you enter, you can't get out.