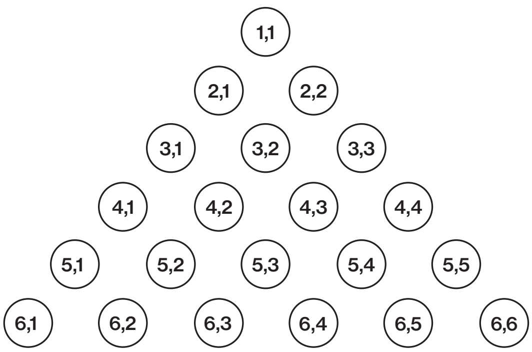
I normally review the ground rules the first issue of each academic year. However, last issue was crowded, so the ground rules were deferred. Because this issue is also crowded, I refer you to the ground rules on the Puzzle Corner website (cs.nyu.edu/~gottlieb/tr/).

Problems

N/D 1. Our wink master, Rocco Giovanniello, offers us the six-layer triangular problem shown below. Initially the 1,1 position is empty and the other 20 positions each have a wink. Moves are diagonal or horizontal jumps, with the jumped-over wink removed. The goal is to find a sequence of 19 jumps so that the remaining wink is in the 1,1 position.



N/D 2. Sorab Vatcha has a sphere of radius R and wonders what is the largest-volume cylinder that can fit inside.

N/D 3. The arithmetic mean of a_o and b_o is of course $(a_o+b_o)/2$, and the geometric mean is $\sqrt{a_o b_o}$. When $a_o = 2$ and $b_o = 20$, we get 11 and $\sqrt{40} \approx 6.832$. Greg Schaffer didn't like either answer; he felt a "reasonable" value would be "something more like between 8 and 9." He tried iterating the procedure defining

$$a_{i+1} = (a_i + b_i)/2 \text{ and } b_{i+1} = \sqrt{a_i b_i}$$

and the values seem to converge to around 8.5, meeting his intuition. He asks whether both sequences always converge to a common limit, and if so, what is that limit when $a_o = 1$ and b_o is large?

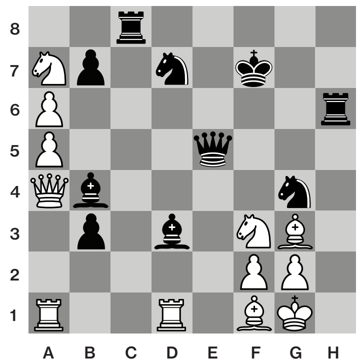
Speed Department

John and Will Marshall write that John told his wife, Jane, that if she tripled the number of days they have been married and subtracted one, she would get a perfect square. Jane disagreed. Who's right?

Solutions

J/A 1. In a refinement of Duncan Ewing's question about the number of available moves in chess, we seek a reachable position from which the number of legal moves is maximal.

Greg Muldowney writes that the maximum number of moves for chess pieces on an unobstructed board are as follows: king 8, queen 27, rooks 14, bishops 13, knights 8, and pawns usually 1 but occasionally 2, 3, or 4. This gives an upper bound of 8 + 27 + 2(14) + 2(13) + 2(8) + 8(2) = 121 moves at one turn. The typical count of available moves is far less: in the opening, when the center of the board is clear, all pieces are highly constrained; through the midgame as pieces become more mobile, pawn structure partitions the board, plus at least one high-mobility piece may have been captured. Therefore the maximal number of moves would likely be found in a midgame after an unusually high count of pawn captures, but where all other pieces remain on the board in positions that minimally constrain, in particular, the queen and rooks. Such a case is illustrated below.



While not an example of expert play (and arguably a bit contrived), this is a reachable position that has 95 possible moves for Black: king 8, queen 26, rooks 14 each, bishops 11 and 9, knights 5 and 4, and pawns 4 in total—including an instance of three options for moving the same pawn. In the context of computing, forward prediction from such a position is complicated by not only the high count of possible moves but also the number of pieces in a state of incipient engagement.

Online references cite similar positions—minimal pawn structure, nearly full mobility of the queen and rooks, captures at or near the edge of the board—with counts of 100 to 105. Another variant of the question, not addressed here, includes multiple promotions, giving rise to legal chess positions with numerous queens on the board and over 200 possible moves.

John Chandler produced a solution with multiple queens on the board and 162 possible moves that I have placed on the Puzzle Corner website (cs.nyu.edu/~gottlieb/tr/).

I have also placed a truly wild solution from Jim Larsen on the website. Larsen's position has 324 legal moves!

J/A 2. Several authors used the binomial distribution to solve John Urschel's probability puzzle. The following example, from Timothy Maloney, also includes an explanation for why the central limit theorem assures that the overall probabilities will approach 0.5 as n increases even if the initial probabilities are far away.

I have put on the website a solution from an author whose name I have lost. I have no doubt the name was on the e-mail, but I printed only the PDF attachment, which is nameless. If the author contacts me, I will correct my error, for which I apologize.

Maloney's solution follows.

Since the cards are withdrawn and replaced in the deck, this is equivalent to a biased coin-toss problem. A binomial distribution applies, with $p = 5/13$ (odd cards) and $q = 8/13$ (even cards). The terms of the distribution are, as usual, $f(x) = C(n,x)\, p^x q^{n-x}$.

What is unique about this problem is that terms with x odd represent odd sums, and those with x even represent even sums. Thus the total probabilities of odd and even are "interspersed" sums of terms. If a minus sign is assigned to the odd terms, a probability function expressing the difference between even and odd probabilities results, $g(x) = C(n,x)(-1)^x p^x q^{n-x}$. The sum of the $g(x)$ terms is the total difference between even and odd probabilities, $\Delta(p,q)$, but note that just as the sum of the $f(x)$ terms is $(p + q)^n = 1$, the sum of the $g(x)$ terms is $\Delta(p,q) = (q - p)^n$. Thus as n grows larger, these sums tend to balance out and approach 0.5 for each probability, with even sums slightly larger than odd for this case of $p = 5/13$. The even/odd probabilities are $0.5 \pm (q - p)^n/2$, respectively. Let's look at the cases cited in Urschel's problem statement.

For $n = 2$, the combinatorial coefficients are 1-2-1, and only the 2 term represents an odd sum, 0.47337 ... Even probability is the complement, 0.52662 ... For $n = 4$, the coefficients are 1-4-6-4-1 and the 4 coefficients are odd. Now the odd sum is 0.49858 ... and its even complement is 0.501418 ...; closer to 0.5 each. The quantity 3/13 is being raised to the powers $n = 2$ and 4, as above, to get these results. For $n = 100$, we get $0.5 \pm 1.0393 \ldots x10^{-64}$ for the odd/even probabilities—really hard to calculate on a spreadsheet with separate terms, or any other way than with the above $q - p$ formula!

Note that the convergence to even/odd probability of 0.5 for large n is a consequence of the central limit theorem, as the binomial distribution approaches a symmetric Gaussian distribution (statistics books suggest this is a good approximation for $np \geq 5$, as certainly is the case for $n = 100$). Note that smaller values of p (such as 1/52, with one ace in a deck of otherwise even-numbered cards) will still give convergence to 0.5 with n large enough.

J/A 3. The following solution to Frank Rubin's problem about mathematicians playing a numbers game is from Daniel Briggs.

One quickly realizes that the simplest ways for such an otherwise strange situation to occur are for all three terms of the sum to always be equal powers of three, or for all three terms of the sum to always consist of a power of two and the power one less, twice. Investigating the powers of three yields no solution; the powers of two work out.

Carefully estimating (for example, using $2^{10} = 1.024 \times 10^3$), one can see that 2^{66} is a 20-digit number, 2^{70} is a 22-digit number, 2^{86} is a 26-digit number, and 2^{112} is a 34-digit number.

Thus we have the solution 8, 32, 4, 16, as

$$(2^3)^{22} = (2^5)^{13} + (2^2)^{32} + (2^4)^{16}$$
$$(2^5)^{14} = (2^3)^{23} + (2^2)^{34} + (2^4)^{17}$$
$$(2^2)^{43} = (2^5)^{17} + (2^4)^{21} + (2^3)^{28}$$
$$(2^4)^{28} = (2^3)^{37} + (2^2)^{55} + (2^5)^{22}$$

Given these numbers, verifying the uniqueness constraints given by the mathematicians is very easy. Proving that this is the only solution would probably be very difficult.

Better Late Than Never

J/F 2. Michael Gordy, Brian McCue, and the proposer independently noted that this problem is more difficult that the solution given suggests. McCue believes that Rhodes's solution (on the website) would be "more than adequate for practical work." Gordy notes that these problems are now known as scan statistics and recommends that interested readers browse https://ia601901.us.archive.org/11/items/bstj37-1-83/bstj37-1-83.pdf.

M/A 3. Ted Mita notes that the published solution minimizes the total cost rather than the number of jars and coins as requested.

J/A SD. My mother was born in Italy, and I have visited bella Roma maybe a half-dozen times in the last decade. As a result, I was embarrassed to receive *many* corrections to the given solution from readers who objected to our use of non-Roman Roman numerals. As noted by Shirley Wilson, there are *rules* for these numerals, and the solution MID doesn't satisfy them. So the largest English word reported is MIX=1009. Readers also noted that if you are willing to play fast and loose with the rules, MIMIC=2098 would be better than the published MID=1499.

Other Responders

Responses have also been received from D. Athanis, M. Brill, P. Chin, J. Conway, D. Dewan, E. Friedman, J-P. Garric, J. Glaser, S. Golson, S. Harada, A. Hirshberg, D. Hudgings, P. Kramer, W. Lemnios, J. Mackro, P. Manglis, N. Markovitz, B. McCue, D. Micheletti, R. Morgan, S. Nason, A. Ornstein, P. Paternoster, S. Peters, J. Prussing, B. Rhodes, K. Rosato, H. Sard, S. Shapiro, T. Sim, B. Sutton, J. Vaughan, C. Viehland, B. Weggel, and J. Wise.

Proposer's Solution to Speed Problem

Jane. Either n^2 or $n^2 - 1$ must be divisible by 3, so $n^2 + 1$ isn't.

Send problems, solutions, and comments to Allan Gottlieb, New York University, 60 Fifth Avenue, Room 316, New York, NY 10011, or to gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.