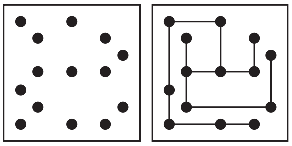
I write this a week before autumn ends, and here in the northeast United States, it has been spectacular. Unusually mild for many weeks, the entire season lacked any major storm so the foliage remained on the trees for all to enjoy. We live on a lake and my office looks out to the lake and the hillside beyond, which this year was filled with orange and yellow trees for over a month.

Yesterday it turned cold; let's see what winter brings. By the time this arrives in your mailbox, spring should be in plain sight.

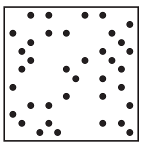
## Problems

**M/A 1.**



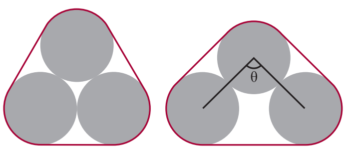
We begin with Yashi, another puzzle from Frank Rubin's website sumsumpuzzle.com. As shown in the $7 \times 7$ example above, you are given a collection of dots situated on an $n \times n$ integer grid and seek a solution using horizontal and vertical lines to connect all of the dots without any crossings. As illustrated in the solution above, there must be exactly one path connecting any dot to any other. More information, other examples, and useful tips can be found on the website.

Rubin offers us the following $14 \times 14$ challenge.



**M/A 2.** Joe Horton and Marcelino Gorospe report that the following problem was an offshoot of their designing a series of devices for treating brain aneurysms. The left diagram below shows three gray mutually tangent circles of radius $r$ enclosed in a red band. (In the medical device the circles are cross-sections of three thick wires and the band is a thin wire.) What is the length of the band in terms of $r$?

In the right diagram the circles have moved apart so the bottom two are no longer tangent. To be specific, we show the angle $\theta$ (which, by symmetry, was $\pi/3$ before) and ask for the band length in terms of $r$ and $\theta$.




**M/A 3.** Eric Schonblom knows a woman who despises pennies (especially the silver Lincoln cents) to such an extent that she never carries any and is normally unwilling to purchase any item for which change must include at least one penny.

Recently this woman emptied some newly minted U.S. coins onto a counter that has glass jars containing candy. Each sort of candy has a different price, a positive integer number of U.S. cents. The coins she emptied on to the counter would permit her to buy two candies from any of the jars, but in each case she would need to supply an additional penny. She also has the right amount to pay for three candies from any jar, but in this case she would need to receive an unwanted penny in change.

In despair she gives up and asks for one candy from each jar, expecting the worst. However, she is pleasantly surprised to find that her coins permit her to pay for the purchase with no change back.

How many jars are there, what is the price of the candy in each, and what coins did she have in her purse? The preferred solution is the one with the fewest jars and the fewest coins.

## Speed Department

Roy Schweiker (with help from his papa, John) is preparing a circular multi-topping Italian delight. He lives in a tiny apartment so is worried about the space required. He knows the radius of the dough is $Z$ and the thickness of the dough is $A$. What is its volume?

## Solutions

**N/D 1.** Another "Texas Hold'em" poker problem from Richard Morgan. In this poker game a number of players—six, in this case—are each dealt two so-called hole cards, and there are five other "community cards" that all six players can use. Thus, each player has access to seven cards (two hole, five community), and each chooses five of them as a hand.

Morgan wants to know the odds that one player's hand contains four nines and another player's hand contains a nine-high diamond straight flush. This was the basis of a promotion at a local casino. That casino required each hand to include both of the player's hole cards. I am asking for the odds both with and without this last requirement.

Three readers responded to this problem, all with fine solutions; Bruce Heflinger's and Ted Mita's appear on the Puzzle Corner Web page and Jerry Grossman's is printed here. Jerry uses the common convention that C(n,p) = n!/((n-p)! × (p!)).

For the version in which the player must use both hole cards, I have the following analysis. Assume the players have names. There are C(52,5) × C(47,2) × C(45,2) × C(43,2) × C(41,2) × C(39,2) × C(37,2) = 1,016,376,286,576,073,046,624,000 ways to lay out the deal; this is the denominator of our probability. In order to meet the conditions, the table must contain the nine of diamonds, one other nine (three ways to choose this), two of the remaining cards in the straight flush (C(4,2) = six ways to choose these), and one other card (44 ways to choose this). There are six ways to choose the player with four nines, and five ways to choose the player with the straight flush (their cards have already been determined by the choices for the table). Then there are C(43,2) × C(41,2) × C(39,2) × C(37,2) ways to choose the cards for the other four players. Thus the numerator of the fraction is 3 × 6 × 44 × 6 x 5 × C(43,2) × C(41,2) × C(39,2) × C(37,2) = 8,682,413,717,577,600. The fraction reduces to 1/117,061,490.

For the version in which the hole cards need not be used, there are two more cases. In the first, the table shows three nines including the diamond nine, together with two of the other four cards in the straight flush, and the player with four nines has a useless card. The calculation of the numerator for this case is 3 × C(4,2) × 6 × 5 × 44 × C(43,2) × C(41,2) × C(39,2) × C(37,2), the same 8,682,413,717,577,600 as before. In the second case, the table has the nine of diamonds, one other nine, and three of the other four cards in the straight flush, and the player with the straight flush has a useless card. The calculation of the numerator for this case is 3 × C(4,3) × 6 × 5 × 44 × C(43,2) × C(41,2) × C(39,2) × C(37,2), which is 2/3 of the previous answer, or 5,788,275,811,718,400. The numerator for this case therefore is 2 × 8,682,413,717,577,600 + 5,788,275,811,718,400 = 23,153,103,246,873,600, and the final probability in this case reduces to 4/175,592,235.

**N/D 2.** Two more "THIRDS" puzzles, these from Geoffrey Coram. Use the letters from the third of the alphabet shown to replace the dashes and produce a standard English word.

–L–P––NT (ABCDEFGH)
––––EY (IJKLMNOPQ)
–AL––– (RSTUVWXYZ)

and

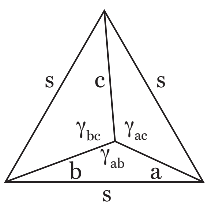
–IR–––– (ABCDEFGH)
D–––H–– (IJKLMNOPQ)
––L–––E (RSTUVWXYZ)

Most responders supplied animal names. Many agreed with Dawn Sapan, who offered "Elephant Monkey Walrus Giraffe Dolphin Vulture." Russell Hendel noted "Airhead" and "Birched" as alternatives; Subramanian Suryanarayanan used "Looney," and Jeremy Katz found "Firebed."

**N/D 3.** Our last regular problem is from John Kotelly, who attributes it to his 1954 Boston Latin teacher Frank Gilbert. What is the side length of an equilateral triangle containing an interior point that is respectively three, four, and five units away from the three vertices?

I received a number of fine solutions, most of which used an iterative technique to solve the resulting equation. Tom Mattick is one of the minority to find a closed form solution, which follows.

The distances between an interior point of an equilateral triangle (side $s$) and the vertices are: $a = 3$, $b = 4$, $c = 5$. Find $s$. A diagram is shown below, including angles $\gamma_{ab}$, $\gamma_{ac}$, and $\gamma_{bc}$.




For triangle $a$-$b$-$s$: $s^2 = a^2 + b^2 - 2ab\cos(\gamma_{ab})$ and similarly for $a$-$c$-$s$ and $b$-$c$-$s$.

For brevity let $K_{ij} \equiv \cos(\gamma_{ij}) = (i^2 + j^2 - s^2)/2_{ij}$, where $i,j = a$, $b$, or $c$. The sum of angles is: $\gamma_{ab} + \gamma_{ac} + \gamma_{bc} = 2\pi$, or $\gamma_{bc} = 2\pi - (\gamma_{ab} + \gamma_{ac})$.

Then $\cos(\gamma_{bc}) = \cos(\gamma_{ab} + \gamma_{ac}) = \cos(\gamma_{ab})\cos(\gamma_{ac}) - \sin(\gamma_{ab})\sin(\gamma_{ac})$, or, in terms of $K$'s: $K_{bc} = K_{ab}K_{ac} - \sqrt{(1 - K_{ab}^2)(1 - K_{ac}^2)}$

Eliminating the radical and simplifying, we obtain:

$$K_{ab}^2 + K_{ac}^2 + K_{bc}^2 - 2K_{ab}K_{ac}K_{bc} - 1 = 0$$

Using equation 1 for the $K_{ij}$'s in equation 2 we obtain a cubic equation in $s^2$, of the form:

$$As^6 + Bs^4 + Cs^2 + D = 0$$

With some algebra, and arranging that $A = 1$, we find:

$$A = 1$$
$$B = -(a^2 + b^2 + c^2) = -50$$
$$C = a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2 = 193$$
$$D = 0$$

Since $D = 0$, we really have a quadratic in $s^2$: $s^4 + Bs^2 + C = 0$, with the solution:

$$s^2 = 25 + \sqrt{25^2 - 193}, \text{ or } s = 6.7664\ldots$$

## Better Late Than Never

**2016 J/A 2.** I must agree with Jay Sinnett, who notes that the problem asked us to construct one piece while the solution constructs two pieces (only one of which moves). I apologize for my misleading wording.

## Other Responders

Responses have also been received from D. August, R. Bird, J. Brown, B. Bruening, T. Chase, D. Diamond, P. Fineman, G. Garmaise, J-P Garric, R. Giovanniello, J. Grossman, B. Heflinger, A. Hirshberg, R. Karla, J. Karlsson, T. Keske, L. Lamel, D. Lampert, W. Lemnios, J. Mackro, Z. Mester, D. Micheletti, T. Mita, J. Mohr, R. Morgen, A. Moulton, E. Nadler, A. Ornstein, T. Palmer, J. Paulsen, J. Prussing, B. Rhodes, K. Rosato, B. Schmolke, M. Seidel, I. Shalom, S. Shapiro, E. Signorelli, J. Sinnett, S. Sperry, A. Stern, M. Strauss, C. Swift, M. Thompson, S. Vatcha, C. Wang, D. Whitman, K. Whitman, R. Whitman, D. Worley, A. Yen, and Y. Zuss.

## Proposer's Solution to Speed Problem

Pi Z Z A

**Send problems, solutions, and comments to Allan Gottlieb, New York University, 715 Broadway, Room 712, New York, NY 10003, or to gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.**