It has been a year since I specified the size of the backlogs for the various kinds of problems that are printed. Currently, I have a large queue of regular and speed problems and a comfortable supply of bridge and other game problems.

Problems

M/J 1. We start with a two-part hearts problem from Matt Lehman.

First a quick review of how hearts is scored. All four players start at zero points. The game ends when at least one player reaches or exceeds 100 points, at which point the winner is the player with the lowest score.

After all tricks have been played, the players examine the tricks they have won and either

i. Each player gets one point for each heart and 13 points also goes to whoever has the queen of spades or

ii. If a player gets all 13 hearts *and* the queen of spades, s/he gets zero points and the other three players each get 26 points.

Lehman asks first, "Is it possible to have a four-way tie, and if so, with what score(s)?" and second, "Is it possible to have a three-way tie, and if so, with what minimal score?"

M/J 2. Edward J. Amrein writes, "On my daughter's birthday a number of years ago, she pointed out that the digits in our two ages were reversed; she was 25 and I was 52. This year, it happened again; she turned 47 and I was 74.

"The fact that this reversal occurred twice got me wondering: Are there certain years for all parents and children when the digits in their ages are reversed? I don't remember it ever happening with my son, who is three years younger than my daughter. I admit that I did not compare our ages for every single year.

"What conditions are required for these age reversals, and when do they occur?"

M/J 3. We end with another two-part problem. This one Christian Erik Kampmann attributes to Martin Gardner.

In the first part, a truth teller asserts that he has (exactly) two children and that one of them is a boy. He asks for the probability that both children are boys. In the second part, the truth teller asserts that he has two children and that one of them is a boy, born on a Tuesday. He again asks for the probability that both children are boys.

Speed Department

Alan Dunwiddie wonders why number theorists were so excited the morning before the last Ides of March.

Solutions

Solutions to the January/February problems are unusually lengthy. They've been shortened to fit in this column, but the full solutions appear on the Puzzle Corner website.

J/F 1. The following solution from Mark Bolotin shows that 15 points is the minimum. First, he offers the hand below to illustrate that 15 points suffices.

♠ 10 9 8 7 6 5 4 3
♥ 10 6 5 4
♦ 4
♣

♠ Q   ♠ K
♥ 9 8 7 6   ♥ K J
♦ Q J 10 9 8 7 6 5   ♦ K
♣   ♣ A K Q J 10 9 8 7 6 5

♠ A J 2
♥ A Q 2
♦ A 3 2
♣ 5 4 3 2

The play is straightforward, but ends with a squeeze. Against a diamond or spade lead, South cashes those aces and the spade jack. He leads a spade to the board, runs his spades, and pitches his clubs and one low diamond. He finesses the king of hearts and cashes both heart honors. Depending on West's discards, there are two possible endings (see full solution online).

In the first scenario, South cashes a diamond and a heart. In the second, South wins two hearts on board. West was squeezed on the last spade.

Is 15 points the lowest possible? Note that a singleton queen and a singleton king on defense lead to the fewest number of high cards for South with a running suit. That is the only way that East-West can have five points in a suit that runs for North-South. Also, North cannot have more than eight of that suit with three or four for South and two defensive singletons. Lastly, North must be able to run a lot of tricks with no points to come up to 13 tricks, since with 4-3-3-3 and fewer than 15 points, South can have very few tricks to run in his hand.

South has to have an ace in any suit West can lead. If South only has two aces, that means that West has 13 cards in those two suits *and* North-South have to run 13 tricks in those suits. However, South can only be 4-3 or 3-3 in those suits; that leaves only 6 or 7 for North. North-South can only play those two suits for at most 10 rounds, not the needed 13. Can South have two suits headed by ace-jack, plus an ace in the third suit? That would mean that East and West have singletons in the two suits, hence 22 cards in the other two suits. But South is either 4-3 or 3-3 in those suits; that adds up to more than 26 cards in two suits. Thus, South has to have at least the equivalent of three aces with a jack in one suit and more than a jack someplace else in his hand—at least 15 points.

J/F 2. John Chandler was pleased to find a search that can be conducted primarily by logic. He writes:

If we require $AB \times C = A \times BC$, then, in particular, we must have the same units digit in both products, and so we know $B \times C - A \times C \equiv 0 \pmod{10}$, i.e., $(B - A) \times C \equiv 0 \pmod{10}$.

Since the digits can't be 0, that means either C is 5 and $B - A$ is even, or $|A - B|$ is 5 and C is even. The first case implies $(10A + B) \times 5 = A \times (10B + 5)$ or $B = 9A/(2A - 1)$. The only integer solutions occur when $2A - 1$ is a divisor of 9, and so the only solutions for A and B are 1,9; 2,6; and 5,5. The last is ruled out because the digits must be distinct, and the second is the example we started with. Thus, we have just one new example from this case: $19 \times 5 = 1 \times 95$.

The other case is actually two cases: $A = B + 5$ or $A = B - 5$. Given $A = B + 5$, we see that $(11B + 50) \times C = (B + 5) \times (10B + C)$ or $B^2 + (5 - C) \times B - 4.5C = 0$.

Thus $B = (C - 5 \pm \sqrt{C^2 + 8C + 25})/2$.

The argument of the square root can be rewritten as $(C + 4)^2 + 9$, which points to the familiar 3-4-5 Pythagorean triple, and so the only value of C giving us an integer square root in this expression is $C = 0$, which is forbidden.

Given $A = B - 5$ we see that $(11B - 50) \times C = (B - 5) \times (10B + C)$ or $B^2 + (-5 - C) \times B + 4.5C = 0$.

Hence $B = (C + 5 \pm \sqrt{C^2 - 8C + 25})/2$.

Here, we get an integer when C is 8 or 4. $C = 8$ gives $B = 9$ or 2.

Since we require A to be positive, the only solution here is $B = 9$, whence we have $49 \times 8 = 4 \times 98$. $C = 4$ gives $B = 6$ or 3. Again $A > 0$ eliminates the latter, yielding only $16 \times 4 = 1 \times 64$.

The reasoning is very similar for finding cases of $A \times BCD = ABC \times D$—i.e., we see that either $|A - C| = 5$ and D is even, or $|A - C|$ is even and D is 5. The full solution online shows that no valid A, B, C, D exists in these cases.

Similarly, for the other two equalities, we get a class of possible solutions for $AB \times CD = A \times BCD$ with $|B - A|$ even and $D = 5$, and for $AB \times CD = ABC \times D$ with $|B - C|$ even and $D = 5$. Two solutions exist: $13 \times 25 = 1 \times 325$ and $39 \times 75 = 3 \times 975$ (details online).

Things get more complicated in the other cases, where $D \neq 5$. Indeed, so complicated that I don't see any refreshing logic to home in on the possible solutions. Hence, I retreat to the exhaustive search and find that there is indeed another solution in this case: $27 \times 56 = 2 \times 756$.

J/F 3. Andrew and Tim Soncrant generalized the problem; their solution appears online. David Detlets, whose solution follows, shows that it is indeed possible to have "too much of a good thing."

Let $S(c, n)$ be the stake of a player using constant c after n coin flips. Then $S(c, n + 1) = S(c, n) \times ((1 - c) + c \times [2 \text{ or } 0.4])$. The second factor is either $1 + c$ or $1 - 0.6c$.

Assume first that n is a large even number. There are $n + 1$ outcomes, whose probabilities are determined by a binomial expansion. In the most probable outcome ($n/2$ heads and tails), the new stake is $S(c, n) = S(c, 0) \times ((1 + c)(1 - 0.6c))^{(n/2)}$.

This is maximized when $c = 1/3$—i.e., $c_{max} = 1/3$ wins for equal numbers of heads and tails. For outcomes with fewer heads than tails, there are some number of $((1 + c)(1 - 0.6c))$ "paired" factors, and some number of excess $(1 - 0.6c)$ factors. For outcomes with more heads than tails, we have excess $(1 + c)$ factors. The total probability of these sets of unequal outcomes are the same. If $c > c_{max}$, then c_{max} wins all the former cases; if $c < c_{max}$, c_{max} wins all the latter cases. Since c_{max} wins the "middle" case, it wins in a majority of cases.

For the case where n is odd, we consider the two middle cases, which have one excess head or tail, and thus one unpaired $(1 + c)$ or $(1 - 0.6c)$ factor. For $c \neq c_{max}$, the ratios $(1 + c)/(1 + c_{max})$ or $(1 - 0.6c)/(1 - 0.6c_{max})$ may exceed 1. However, since c_{max} maximizes $(1 + c)(1 - 0.6c)$, the ratio of this for c_{max} to any other value of c exceeds 1. And this is raised to the power $\lfloor n/2 \rfloor$. As $n \to \infty$, this will exceed the single-term ratio, so, as n gets large, c_{max} wins for both middle terms. As before, c_{max} will win for the rest of the cases with more or fewer heads, so it wins overall.

Obviously, since the expected value of a trial is positive, $c = 1$ maximizes the expected value, but it does not lead to the strategy most likely to dominate.

Better Late Than Never

S/O 1. Scott Nason notes that the problem asked for the "weakest combined holding," which can be interpreted more strictly than "minimal point count." There are solutions with the same point count as the one published but with weaker other cards.

Y2014. E. Signorelli offers $21 = 41 - 20$, with only one operator.

J/F SD. Richard Bair extended the solution to other Platonic solids.

Other Responders

Responses have also been received from F. Albisu, R. Bator, H. Cortina, D. Freeman, J. Harmse, Y. Hinuma, J. Kenton, M. Lawler, K. Lebensold, W. Lemnios, D. Loeb, J. Mackro, S. McGinnis, T. Mita, R. Morgen, A. Muenz, O. Ornstein, M. Piazza, J. Prussing, E. Ramirez, H. Sard, S. Schweighart, I. Shalom, R. Sheffield, A. Shuchat, L. Smith, P. Smoot, L. Stein, W. Stein, C. Van Peski, B. Wake, G. Waugh, and D. Worley.

Proposer's Solution to Speed Problem

Because (using the weird U.S. month/day/year notation) that morning included the time 3/14/15 9:26:53 when they were all measuring the circumferences of half-unit circles.