It's been a year since I reviewed the criteria used to select solutions for publication. Let me do so again.

As responses to problems arrive, they are simply put together in neat piles, with no regard to their date of arrival. When it is time for me to write the column in which solutions appear, I first weed out erroneous and illegible responses. For difficult problems, this may be enough; the most publishable solution becomes obvious. Usually, however, many responses still remain. Next I try to select a solution that supplies an appropriate amount of detail and that includes a minimal number of characters that are hard to set in type. A particularly elegant solution is, of course, preferred, as are contributions from correspondents whose solutions have not previously appeared. I also favor solutions that are neatly written, typed, or sent by e-mail, since these simplify typesetting.

Problems

N/D 1. Sorab Vatcha would like you to construct a 9-by-9 sudoku puzzle for the fewest seed numbers required to a unique solution, and then solve it.

N/D 2. Steve Silberberg views life not as a bowl of cherries but, instead, as a four-dimensional integer lattice. We are located at the origin and wish to get to (x, y, z, t) with all four positive integers. Each individual move is to advance one unit *forward* in one dimension. How many different paths are possible as a function of the target (x, y, z, t)? For example, two of the legal paths to (1, 1, 1, 1) would be

$$(0, 0, 0, 0) \rightarrow (0, 0, 1, 0) \rightarrow (0, 1, 1, 0) \rightarrow (0, 1, 1, 1) \rightarrow (1, 1, 1, 1)$$

and

$$(0, 0, 0, 0) \rightarrow (0, 0, 0, 1) \rightarrow (1, 0, 0, 1) \rightarrow (1, 0, 1, 1) \rightarrow (1, 1, 1, 1)$$

N/D 3. Nob Yoshigahara, honoring the late Japanese cryptogramist Kyoko Ohnishi, offers one of her masterpieces. As usual, each letter stands for a unique digit.

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   LOVE
 ×   IS
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  □□□□□
 □□□□
 ------
  BLIND
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Speed Department

Dan Diamond wonders what size cube has the same number of square inches in its surface area as it has cubic inches in its volume. What about spheres? Surprise!

Solutions

J/A 1. Tom Hafer created a three-dimensional sudoku-like problem he calls "wordoku," involving four horizontal levels. His instructions were as follows.

"This is a 4-by-4-by-4 cube with 16 different symbols. It is solved exactly like a normal two-dimensional sudoku problem except that each symbol can occur only once in each cardinal plane, including the vertical dimension. Also, on this one a bit of intuition is required. When the solution is complete, a message or pattern will emerge."

First, I must apologize to Mr. Hafer for misspelling his name in the July/August issue. A solution comes from Aaron Ucko.

Comparing the third and fourth levels shows each (known) symbol moving one space down and to the right, wrapping around in each dimension. Extrapolation yields a consistent second layer of

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UZLP
MIT4
ONKW
ASYE
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and a full solution of

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EASY   UZLP   IT4M   KWON
PUZL   MIT4   NKWO   YEAS
4MIT   ONKW   SYEA   LPUZ
WONK   ASYE   ZLPU   T4MI
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The first layer gives the message

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EASY   PUZL   4MIT   WONK
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or "Easy puzzle for MIT wonk."

J/A 2. Warren Smith's interest in improved voting schemes for democracies extends to puzzles on voting and related topics, including the following.

Suppose there are four dice: Blue, Green, Red, and White. These dice have different numbers than usual printed on their six faces. After observing a long sequence of experiments rolling pairs of these dice, you conclude the following:

- When both are rolled simultaneously, the blue die gives a higher number than the green die two-thirds of the time.
- When both are rolled simultaneously, the green die gives a higher number than the red die two-thirds of the time.
- When both are rolled simultaneously, the red die gives a higher number than the white die two-thirds of the time.

You are now asked to consider rolling the blue and white dice simultaneously. What can you conclude about the probability P that the blue die will produce a higher value than the white one?

Donald Aucamp was able to find a solution with $P = 1/3$, which appears to be the minimum possible. Note that information on voting schemes, including this and other puzzles, can be found on Smith's website rangevoting.org. Aucamp writes:

> In the interest of clarity, I changed the dice descriptions from colors to A, B, C, and D. We are given that $P(A > B) = 2/3$ is the probability that A exceeds B when these two dice are rolled together; similarly, we're told that $P(B > C) = P(C > D) = 2/3$ and must determine what can be said about $P(A > D)$. For convenience, and

with no loss in generality, assume the 36 faces are all different and are numbered from 1 to 36. Define the six possible outcomes for A to be $A_1 < A_2 < A_3 < A_4 < A_5 < A_6$, and similarly for B, C, and D.

From Bayes' Law:

1) $P(A > B) = \sum_{i=1}^{6} P(Bi) \times P(A > B|B = B_i) = \sum (1/6)(n_i/6) = N(A > B)/36$ where n_i is the total number of A outcomes that exceed B_i and $N(A > B) = \sum n_i$. For example, if the numbers on A are 1, 4, 5, 9, 22, and 36, and if $B_i = 7$, then three of the As exceed 7 and n_i=3. Since $P(A > B) = 2/3$, it is required from (1) that $N(A > B) = 24$. Similarly, $N(B > C) = N(C > D) = 24$. Thus, an equivalent problem is to find the possible values of $N(A > D)$, given that $N(A > B) = N(B > C) = N(C > D) = 24$.

Consider the ordering given by (2) below, where the letter values increase from 1 to 36:

2) C_1C_2 $D_1D_2D_3D_4D_5D_6$ A_1A_2 C_3C_4 $B_1B_2B_3B_4B_5B_6$ C_5C_6 $A_3A_4A_5A_6$

From the assumed convention, note that $C_1 = 1$, $C_2 = 2$, $D_1 = 3$, etc. The ordering in (2) is feasible since $N(A > B) = N(B > C) = N(C > D) = 24$. To see this in the case of $N(A > B) = 24$, note that A_1 and A_2 are smaller than the Bs, and A_3, A_4, A_5, and A_6 are greater, so $n_1 = n_2 = 0$ and $n_3 = n_4 = n_5 = n_6 = 6$. Note also that all the As exceed all the Ds, so $N(A > D) = 36$ and $P(A > D) = N(A >vD)/36 = 1$.

An important characteristic of (2) is that the six Bs are grouped together. Feasibility requires four As to the right of the Bs, two As to the left, two Cs to the right, and four Cs to the left. Grouping the Bs in this manner maximizes the freedom of movement of the As and Cs. The four As, for example, can be placed anywhere to the right of the Bs. This overall freedom of movement makes it easy to move the letters to achieve the total spectrum of possible values of $N(A > D)$. Using this movement technique to achieve the minimum value of $N(A > D)$ results in the following:

3) A_1A_2 $D_1D_2D_3$ $C_1C_2C_3C_4$ $B_1B_2B_3B_4B_5B_6$ $A_3A_4A_5A_6$ $D_4D_5D_6$ C_5C_6

Since $N(A > D) = 12$, then $P(A > D) = 12/36 = 1/3$. Note that no more feasible moves reducing $N(A > D)$ are possible. Since all the exchanges in going from 36 to 12 could have been made incrementally, the final answer to the problem is $P(A > D) = n/36$, where n is any integer satisfying $12 \le n \le 36$. The limits of $P(A > D)$ are therefore given by $1/3 \le P(A > D) \le 1$.

J/A 3. Victor Barocas teaches a class in which students sit in rows. When he gives an exam, he asks students not to leave early unless they can do so without having to disturb anyone else—that is, unless there is an unobstructed path to a side of the room. As a result, a bunch of students sometimes build up and are then "released" when a student on one end finishes. Obviously, when the second-to-last student in a given row finishes, everyone in that row can leave except for the last student, because the row has two ends. If the row has N students in it, what is the expected number of students to leave when the second-to-last student finishes?

Ed Sheldon attacked the problem by first noting that there are $N - (i + 1)$ pairs of seats that have exactly i seats in between them. All these pairs would account for $i(N - i - 1)$ students being released. This observation gives rise to the following table.

# between	# of pairs	# released
0	$N - 1$	$0(N - 1) = 0$
1	$N - 2$	$1(N - 2) = N - 2$
2	$N - 3$	$2(N - 3)$
...		
$N - 3$	2	$(N - 3)2 = 2(N - 3)$
$N - 2$	1	$(N - 2)1 = N - 2$

The total number of pairs is the sum of the middle column, which is clearly $N \times (N - 1)/2$. The total number released is the sum of the third column, which is (not so clearly, perhaps) $(N - 1)2 \times (N - 2)/2 - (N - 2) \times (N - 1) \times (2N - 3)/6$.

The total number released divided by the total number of pairs is the expected number of students released, which is $(N - 1) \times (N - 2)/N - (N - 2) \times (2N - 3)/(3N) = (N - 2)/3$.

Finally, the number of students leaving is one more than the above, to account for the second-to-last student.

Dan Katz arrives at the same answer by a rather different technique. He reformulates the problem as a circle with $N + 1$ slots, in which he randomly places two red pegs and one black peg. The two red pegs represent the last two students to finish, and the black peg is a placeholder that separates the beginning of the row of students from the end of the row.

Thus, the expected number of trapped students in the row is the same as the expected number of empty slots between the two red pegs. There are $N - 2$ empty slots, and by symmetry, each empty slot has an equal chance of falling into any of the three regions between pegs. This means each of the $N - 2$ slots has a one-third probability of counting as a trapped student, and thus the expected number of trapped students is $(N - 2)/3$.

Better Late Than Never

2011 N/D 3. Dan Sidney's updated solution is on the website (cs.nyu.edu/~gottlieb/tr).

2012 M/A 2. Jeffrey Schenkel found an additional solution.

M/J 3. Chris Connell found a simpler solution.

J/A SD. Sorab Vatcha notes that we should have said distinct palindromes.

Other Responders

Responses have also been received from J. Feil, C. Larson, W. Lemnios, A. Ornstein, D. Plass, J. Rible, K. Rosato, P. Schottler, E. Sheldon, E. Signorelli, and cadanau@alum.mit.edu.

Proposer's Solution to Speed Problem

Edge size = diameter = six inches.

Send problems, solutions, and comments to Allan Gottlieb, New York University, 715 Broadway, Room 712, New York, NY 10003, or to gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.