As I write this column in early April, spring has finally arrived in the northeastern United States. We have apparently been spoiled by a string of what I hear are called "Al Gore winters," and as a result, this one has seemed historically harsh. But in fact we have had much worse. I remember one winter as a Brandeis graduate student when two consecutive weekends (or perhaps two out of three) had snows of 24 and 30 inches. I also remember a low of –20 °F, which I recall was significantly harsher than the –10s we had already encountered.

Flash: we are critically low on speed problems.

PROBLEMS

J/A 1. Larry Kells has his minimalist hat on again: he wants to know the minimum number of high-card points (counting only A–J, no 10) that North-South can hold and still make 3 no trump. What about 6 no trump? (Larry feels 7 is too easy.)

J/A 2. Robert Gilroy writes that the current interest in Sudoku reminds him of a similar puzzle he encountered in the 1950s. In this problem we have an eight-by-eight grid into each square of which we are to place the digits 1 through 8 so that no number is repeated in any row, column, or diagonal (the latter do not wrap around, so most diagonals have fewer than eight squares). Robert has a solution where two squares are left blank, but he's hoping one of you can find a full solution, or at least one in which only a single square is blank.

J/A 3. Richard Hess offers the following challenge. You are to use the digits 2, 5, 6, and 9 once each to form a mathematical expression for each integer from 0 to 128. The rules are somewhat different from those of our "yearly problem": You may use addition, subtraction, multiplication, division, powers, decimal points, concatenation (2 and 5 combine to make 25), and parentheses. No roots, factorials, or other notations may be used.

SPEED DEPARTMENT

John Astolfi wants you to play "name that number." My multiplicative inverse is also my additive inverse. Who am I?

SOLUTIONS

M/A 1. Somehow, during the production process, West was given the same spade holding as East. This did not slow down our readers, who presumably considered correcting the typo "part of the problem." For example, we have the following solution from Steve Shalom.

"I assume that there's a typo in the problem as given, since West has only 11 cards, with a spade holding the same as East's. Changing West's spade holding to KJ97 makes the deal consistent.

"Then, South can take all the tricks at 7 no trump as follows:

Trick 1: ♣8, ♣9, ♣10, ♣J
Trick 2: ♣7, ♣6, ♦2, ♠A
Trick 3: ♣5, ♣4, ♦3, ♠Q
Trick 4: ♠2, ♠9, ♠10, ♥A
Trick 5: ♠8, ♥K, ♠3, ♠7
Trick 6: ♥9, ♥7, ♥10, ♥8
Trick 7: ♥6, ♠K, ♠6, ♥5
Trick 8: ♥4, ♠J, ♦4, ♥3
Trick 9: ♠5, ♣A, ♦5, ♥Q
Trick 10: ♠4, ♣K, ♦6, ♥J
Trick 11: ♥2, ♣Q, ♦7, ♦A
Trick 12: ♣3, ♦K, ♦8, ♦Q
Trick 13: ♣2, ♦10, ♦9, ♦J

"Of course, trivially different solutions can be found by switching equivalent cards or by switching tricks 2 and 3, 4 and 5, or 7 and 8.

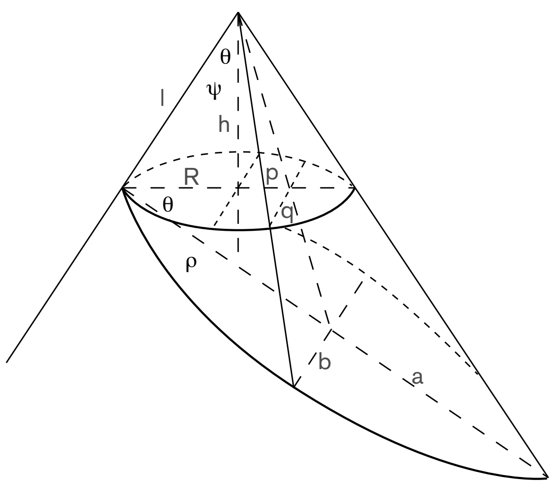
"The North-South hands can be made even weaker and still take all 13 tricks: switch the 8 and 9 of clubs and the 8 and 9 of hearts."

M/A 2. Everyone agrees with Don Dechman that the first 10-digit prime occurs surprisingly early in the expansion of π. It starts with the fifth digit: specifically, 5,926,535,897. If anyone finds a 100-digit solution, please let me know.

M/A 3. I received a number of beautiful solutions to this problem. Hendrata Dharmawan's can be found at dharmath.blogspot.com/2010/03/solution-osculating-circle-ellipse-and.html. Here are two more, the first of which is from Henri Hodara.

The parametric expression for an ellipse with semi-major and minor axes a and b is (1) $x = a\cos t, y = a\sin t$.

From elementary calculus, the radius of curvature is (2) $\rho = (x'^2 + y'^2)^{3/2}/(x'y'' - y'x'')$; ' and '' denote first and second derivatives with respect to t. From (1) and (2), the radius of curvature of



the osculating circle, at the "matching point" where the elliptical curve intersects its major axis, is (3) $\rho = b^2/a$.

Refer to the right circular cone of half-angle θ. Draw a plane perpendicular to the cone axis and passing through the point where the major axis of the ellipse intersects the generatrix of the cone. That intersection is a circle of radius R. For the center of the "major-axis osculating circle" to lie on the cone axis, its radius of curvature must be

(4) $\rho = R/\cos\theta$.

Thus it remains for us to show that (3) = (4); in other words,

(5) $b^2/a = R/\cos\theta$.

In order to prove (5), we need to express a and b in terms of R and θ.

Refer now to the circle of radius R. Call h the perpendicular from the cone vertex to center of the circle, l the length of the cone generatrix between its vertex and that circle, p the radial distance between its center and the point where the straight line joining the cone vertex and the ellipse center intersects the plane of that circle, and $(\psi - \theta)$ the angle between that line and h.

From the geometry of the figure, the major axis of the ellipse is

(6) $a = R/\cos\theta/(1 - \tan^2\theta)$.

To find the minor axis b, we need to calculate the ratio b/q, where q is the intersection in the plane of the circle that results from projecting b from the cone vertex. Referring to the figure,

(7) $b^2/q^2 = ((l/\cos\psi)/(h/\cos(\psi - \theta)))^2 = (\cos(\psi - \theta)/\cos\psi\cos\theta)^2$

(8) $q^2 = R^2 - p^2 = R^2[1 - \tan^2(\psi - \theta)/\tan^2\theta]$.

Combining (6), (7), and (8), and noting that

(9) $\tan\psi = a/(\sin\theta/R) = \tan\theta/(1 - \tan^2\theta)$, we get

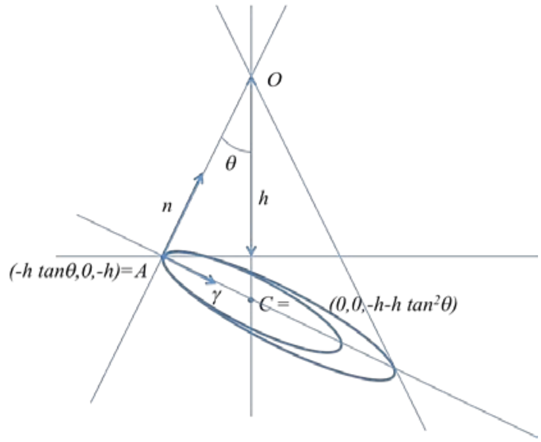
(10) $b^2/a = R\cos\theta(1 - \tan^2\theta)(\cos(\psi - \theta)/\cos\psi\cos\theta)^2[1 - \tan^2(\psi-\theta)/\tan^2\theta]$.

Repeated use of (9) with elementary trigonometric manipulations of sums and differences of angles reduces (10) to the desired expression: $b^2/a = R/\cos\theta$.

The second solution, from Apo Sezginer, begins by choosing the origin to be at the apex of the cone. The equation of the cone is (1) $z^2\tan^2\theta = x^2 + y^2$, where θ is the half-angle of the cone. Let $A = (-h\tan\theta, 0, -h)$ be the point at which the osculating circle matches the ellipse. Without loss of generality, we select the coördinate axes such that A is in the $y = 0$ plane. The plane that defines the conic section goes through A, and the plane is orthogonal to the unit vector $\hat{n} = (\sin\theta, 0, \cos\theta)$ along $\overrightarrow{AO}$. The equation of the plane is (2) $(x + h\tan\theta)\tan\theta + (z + h) = 0$.

The parametric equation of the intersection of the cone and the plane is (3) $r: [0, 2\pi] \to \Re^3$; $r(u) = \Big(-ht + (1 - \cos u)(ht/1 - t^2), \sqrt{(1 + t^2)/(1 - t^2)}ht\sin u, -h - (1 - \cos u)ht^2/(1 - t^2)\Big)$.

We use the shorthand notation $t = \tan\theta$. According to the original hypothesis, $1 - t^2 > 0$, as the conic section would be a parabola for $t^2 > 1$. Parametric equation (3) of the conic intersection is obtained by substituting an affine transformation of $(\cos u, \sin u)$ for (x, y, z) in the equations of the cone (1) and the plane (2), and matching the coeficients of the resulting Fourier series with respect to the parameter u. The curve maps 0 to $r(0) = A$. The curvature vector at A is:



$(d^2r/ds^2)\big|_{u=0} = \ddot{r}/(\dot{r}\cdot\dot{r}) - \dot{r}\,[(\ddot{r}\cdot\dot{r}/(\dot{r}\cdot\dot{r})^2]\big|_{u=0} = (1, 0, -t)/ht(1 + t^2) = [(1, 0, -t)/\sqrt{1 + t^2}]\,[1/(ht\sqrt{1 + t^2})] = N\kappa$

Above, dr/ds denotes derivative with respect to path length and $\dot{r} = dr/du$ denotes derivative with respect to the parameter u. The unit normal $N = (1, 0, -t)/\sqrt{1 + t^2}$ is in the direction $\overrightarrow{AC}$ where C is the center of the osculating circle. The curvature $\kappa = 1/(ht\sqrt{1 + t^2})$ is the reciprocal of the radius of the osculating circle, $R = ht\sqrt{1 + t^2}$. The center C of the osculating circle is in the direction of N, a distance R from A:

$C = A + NR = A + [(1, 0, -t)/\sqrt{1 + t^2}]\,ht\sqrt{1 + t^2} = (-ht, 0, -h) + (ht, 0, -ht^2) = (0, 0, -h - ht^2)$

Therefore, the center of the osculating circle is on the axis of the cone, at $C = (0, 0, -h - ht^2)$.

BETTER LATE THAN NEVER

2009 N/D 3. Andrew Moulton notes that the revised solution on the website has a typo. Case 3 should begin "If $15 \leq W1$ ...."

2010 M/A SD. As several readers note, the fair coin is indeed more likely, but each probability should be multiplied by $\binom{10}{4} = 210$.

OTHER RESPONDERS

Responses have also been received from F. Albisu, Aurion, S. Berger, G. Blondin, R. Byard, P. Cassady, D. Dreyfuss, D. Goldfarb, R. Hess, R. Krawitz, A. LaVergne, W. Lemnios, M. Lieberman, J. Marshall, F. Model, C. Morton, E. Nelson-Melby, F. Pollitz, K. Rosato, E. Sard, A. Shuchat, H. Thiriez, and R. Virgile.

PROPOSER'S SOLUTION TO SPEED PROBLEM

The complex number $i = \sqrt{-1}$. ■

Send problems, solutions, and comments to Allan Gottlieb, New York University, 715 Broadway, Room 712, New York, NY 10003, or to gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.