Our home is on a lake 50 miles north of New York, and the birds have returned this past week (I write this in April)—a welcome sign of spring. The morning birdsong is delightful, and with the trees budding and the rhododendrons ready to pop, May should once again be my favorite month.

My older son lives in San Diego, where the weather is basically beautiful all year. He went to MIT but can't imagine living in such "harsh" conditions again. However, as I point out often (to mostly deaf ears), you cannot appreciate spring without winter.

### PROBLEMS

**J/A 1.** Last year we considered a problem in which North-South makes seven spades, and we wondered what was the most tricks East-West could make in a spade contract (in all cases with best play). Tom Terwilliger asks about a generalization in which we drop the requirement of a grand slam. Again assuming best play, what is the greatest swing in the number of tricks that can occur by having different sides play the hand in the same trump suit? For example, if North-South can make five spades (11 tricks) and East-West can make four spades (three tricks for North-South), the swing would be 11 – 3 = 8.

**J/A 2.** Geoffrey Landis was having dinner with five friends. They all raised a toast and clinked glasses. Since their glasses were all the same diameter, at any instant only three could mutually touch at the rims. For six people, having each touch everybody else's glass requires 15 pairwise touches. Can this be done with five three-glass touches? If not, what is the minimum number required? (This is a 2-D problem; the glasses must touch at the rim.)

**J/A 3.** Nob Yoshigahara sent us this cryptarithmetic problem from Kyoko Ohnishi. Replace each letter with a unique digit to give a true statement.

```
  COLOUR
  COLOUR
  COLOUR
  COLOUR
  COLOUR
  COLOUR
+ COLOUR
 -------
 RAINBOW
```

### SPEED DEPARTMENT

Walter Cluett has a sentence consisting of a one-letter word followed by a two-letter word, then a three-letter word, etc. Can you match or exceed his effort?

### SOLUTIONS

**M/A 1.** Our bridgemeister, Larry Kells, wants you to make seven hearts against best defense despite one opponent's holding the J97543 of hearts, a side ace, and a guarded side king. Oh, yes—the other opponent has 10 high-card points.

Richard Hess was able to improve on the problem: his "other opponent" has 11 high-card points.

| | | |
|---|---|---|
| | ♠ A Q<br>♥ Q 10 8 6<br>♦ *x x*<br>♣ *x x x x x* | |
| ♠ K *x*<br>♥ J 9 7 5 4 3<br>♦ *x x x x*<br>♣ A | | ♠ J *x x x*<br>♥ –<br>♦ K J<br>♣ H Q J *x x x x* |
| | ♠ *x x x x x*<br>♥ A K 2<br>♦ A Q 10 9 8<br>♣ – | |

North is the declarer, and any lead by East permits the first four tricks to be taken as a spade finesse, a diamond finesse, a spade cash, and a club ruff ending in South's hand. He then cashes three diamonds, producing the following (East's hand is immaterial):

| | |
|---|---|
| | ♠ –<br>♥ Q 10 8 6<br>♦ –<br>♣ *x x* |
| ♠ –<br>♥ J 9 7 5 4 3<br>♦ –<br>♣ – | |
| | ♠ *x x x*<br>♥ A K<br>♦ 8<br>♣ – |

The remaining tricks are a high-finessing cross-ruff: spades and diamonds are ruffed by the lowest needed card from North, clubs are ruffed by the A and K by South, and West helplessly underruffs each time.

**M/A 2.** Avi Ornstein (and his friend Fibo) like to play with sequences. Choose an integer $a \geq 2$ and consider the two sequences

$$y_1 = 1 \qquad y_2 = a - 1 \qquad y_n = a \cdot y_{n-1} - y_{n-2} \qquad \text{and}$$
$$x_1 = 1 \qquad x_2 = a \qquad x_n = a \cdot x_{n-1} - x_{n-2}$$

How are these two sequences related?

A number of readers found the relationship $y_n = x_n - x_{n-1}$ experimentally, and several then proved it by induction. Burgess Rhodes and a very few others also solved the difference equations to find the closed form for $x_n$. His solution follows.

"Relationship between $y_n$ and $x_n$. Let $z_n = x_n - x_n - 1$. Then by subtracting recursion $x_n - 1 = ax_n - 2 - x_{n-3}$ from recursion $x_n = ax_{n-1} - x_{n-2}$ we determine that sequence $\{z_n\}$ also satisfies the same recursion: $z_n = az_{n-1} - z_{n-2}$. We may define $x_0 = y_0 = 0$. Then initial conditions for sequence $\{z_n\}$ are $z_1 = 1$ and $z_2 = a - 1$, the same as for sequence $\{y_n\}$. Thus sequences $\{z_n\}$ and $\{y_n\}$ are the same. Terms in the original sequences are related by $y_n = x_n - x_n - 1$.

"Solution for $x_n$. The recursion $x_n - ax_{n-1} + x_{n-2} = 0$ is a second-order, linear, homogeneous difference equation with constant coefficients. By analogy with differential equations of this structure, a solution of the form $x_n = \rho^n$ for some constant $\rho$ is expected. Substituting $\rho^n$ for $x_n$ in the recursion, we obtain $\rho^{n-2}(\rho^2 - a\rho + 1) = 0$ from which determine

$$\rho = \begin{cases} 1 & \text{for } a = 2\text{, a double root, and} \\ \dfrac{a \pm \sqrt{a^2-4}}{2} & \text{for } a > 2 \end{cases}$$

"(1) For a = 2 the general solution is $x_n = C_1 \cdot 1^n + C_2 n \cdot 1^n$. Incorporation of the initial conditions yields $x_n = n, n = 1, 2, \ldots$

"(2) For $a > 2$ the general solution is

$$x_n = C_1\left(\frac{a+\sqrt{a^2-4}}{2}\right)^n + C_2\left(\frac{a-\sqrt{a^2-4}}{2}\right)^n.$$

"Incorporation of the initial conditions yields

$$x_n = \frac{1}{\sqrt{a^2-4}}\left(\frac{a+\sqrt{a^2-4}}{2}\right)^n - \frac{1}{\sqrt{a^2-4}}\left(\frac{a-\sqrt{a^2-4}}{2}\right)^n, n = 1, 2, \ldots \quad \text{(S)}$$

"Notes: Results above do not require that $a$ is an integer. But if it is, then sequences $\{x_n\}$ and $\{y_n\}$ are sequences of integers, as is clear from their definitions as recursions. Thus, while it's not evident, solution (S) is integer for integer $a > 2$ and all $n$. The closed form for the Fibonacci sequence

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

is surprisingly integer for all $n$ as well."

**M/A 3.** We close with another "logical hat" problem from Richard Hess. Recall that in such problems each logician wears a hat with a positive integer on it. All the logicians are error free in their reasoning and are given this information as well as other information in the problem.

Integers $x$ and $y > 2$, but not necessarily different, are chosen. The number on A's hat is $x \cdot y$ and the number on B's is $x + y$. They make statements as follows.

A1: "There is no way you can know the number on your hat."

B1: "I now know my number."

A2: "I now know my number. Both our numbers are less than 500."

What numbers are on A and B?

I found this one rather difficult, and I received several different solutions. I believe Jonathan Hardis is correct in his analysis that $(x,y) = (146, 3)$; A = 438; B = 149. He showed that it is not permitted for $x$ to be prime and $y$ a power of two, which eliminates some of the other answers. Hardis writes:

"From 'A1: There is no way you can know the number on your hat,' we conclude that A reasoned as follows. If you (B) saw a number that was the product of two primes, you could deduce that their sum was on your own hat. Hence, it is not possible to express the number that I'm seeing as the sum of two prime numbers greater than 2. By Goldbach's Conjecture, we both know that all even numbers can be expressed as the sum of two prime numbers. So B is odd, and without loss of generality, $x$ must be even and $y$ must be odd. All factors of 2 in the number you see must be in $x$, so if you saw a number that was $2^n$ times a prime number you could deduce that $y$ was the prime and would know $x+y$ (the number on your own hat). Hence, it is not possible to express the number that I'm seeing as the sum of a prime number and $2^n$.

"This analysis eliminates many small values for B. So now, both logicians know that B must be 127, 149, 193, 251, 253, 331, 337, 373, 403, …

"From 'B1: I now know my number,' we conclude that B reasoned as follows: Of the various ways to factor the number I see, only one gives $(x, y)$ that sums to a number on your list. Now they both know that A must be 372, 438, 492, …

"From 'A2: I now know my number,' we conclude that A reasoned as follows: The number I see (149) is uniquely formed by adding factors of one of the numbers on your list (438). (In contrast, had I seen 127, then my number could have been either 372 or 492.)"

#### OTHER RESPONDERS

Responses have also been received from C. Abzug, S. Berger, M. Brill, C. Coltharp, G. Coram, F. Cornelius, D. de Champeaux, D. Emmes, S. Feldman, H. Fletcher, S. Gordon, O. Helbok, H. Ingraham, P. Kramer, P. Manglis, A. Ornstein, J. Prussing, K. Rosato, P. Schottler, A. Seckinger, M. Seidel, E. Signorelli, C. Swift, T. Turnbull, C. Wampler, D. Watson, J. Wouk, and M. Zeitlin.

#### PROPOSER'S SOLUTION TO SPEED PROBLEM

I am not very happy seeing various obsolete, pointless, uninspired, restrictive, shortsighted, unimaginative, unconventional, interchangeable, misinterpretable reclassifications. ■

---

Send problems, solutions, and comments to Allan Gottlieb, New York University, 715 Broadway, Room 712, New York NY 10003, or to gottlieb@nyu.edu.