Puzzle Corner

INTRODUCTION

As many of you know, MIT has adopted an "open courses" policy. I very much approve of this initiative and sent the message below to President Vest the day it was announced. Naturally, I was pleased to receive a prompt and thoughtful reply.

"I awoke this morning and was delighted to read the *New York Times* article about MIT's new initiative in open courses. I graduated in 1967 (course XVIII) and my son, David, is a freshman. During parents' weekend I was struck by the (to my memory, increased) dedication of the MIT faculty to undergraduate education. I am a professor of computer science at New York University and know a number of MIT computer science faculty so have heard of this emphasis, but it was striking to see it in action. Last semester, David had a question about 18.03 and, by looking on the Web, I could find all of the class material needed to help, and in the process relearn some mathematics and rekindle my lifelong love of the subject.

I have had my computer science courses on the Web off of my home page (allan.ultra.nyu.edu/~gottlieb) for several years and often must explain to friends and colleagues why I would do this and not try to package the material in some commercial manner. Now I can simply respond by noting that 'MIT does it Institute-wide.'

I have never been more proud of my MIT degree than I am today."

PROBLEMS

Jul/Aug 1. Larry Kells wants you to construct a hand in which South can make 5 diamonds despite an opponent holding KJ97543 of diamonds and 14 high-card points overall.

Jul/Aug 2. Ramon Mireles enjoyed this problem from Long's elementary Number Theory. Prove that if p is a prime not 2 or 5, then p must divide infinitely many of the numbers 9, 99, 999, ...

Jul/Aug 3. Roy Sinclair wants you to cut an equilateral triangle with straight line segments into four pieces and reassemble the pieces to form a square. He believes that this was first solved by H. E. Dudeney in 1902.

SPEED DEPARTMENT

My NECI colleague Sanjay Palnikar wants to know the names of the following "standard" conversion factors:

Ratio of an igloo's circumference to its diameter;
2.4 statute miles of intravenous surgical tubing at Yale University Hospital;
2000 pounds of Chinese soup;
10^{-6} mouthwash;
Speed of a turtle breaking the sound barrier;
Time it takes to sail 220 yards at one nautical mile per hour;
16.5 feet in the Twilight Zone;
10^6 aches.

SOLUTIONS

Mar 1. Ken Fan notes that one can cut a square along both diagonals to obtain four congruent triangles and then pair up the triangles along their long sides to obtain two congruent squares. He is interested in the same problem for triangles. Specifically, can one start with an equilateral triangle, cut it into pieces, and rearrange them to form two congruent equilateral triangles? If so, how? If not, why not?

Although I believe the problem is not ambiguous, the wording does seem to have led to different understandings of the requirements. Some believed that the cut lines for the triangle, like those for the square, must be diagonals; this was not intended. Others believed that it was sufficient to rearrange the pieces into equilateral triangles; but the problem requires that the resulting triangles be congruent, not just similar (I should add that with this looser requirement an infinite number of solutions were given).

The following lovely solution is from Craig Wiegert and Adrian Childs. Start with an equilateral triangle of side length 1, which we want to cut and reassemble into two equilateral triangles of side length $a = 1 / 2$. One of the smaller equilateral triangles can be inscribed within the larger, as shown in Figure 1. It is straightforward to calculate that $b = 1/2 - 1 / 2\sqrt{3}$ (using the Law of Cosines) and that $\theta = 15$ degrees (using the Law of Sines).

Next, cut an equilateral triangle of side length b from each of the 15-60-105 triangles, and bisect each of these equilateral triangles. The resulting nine pieces can be rotated

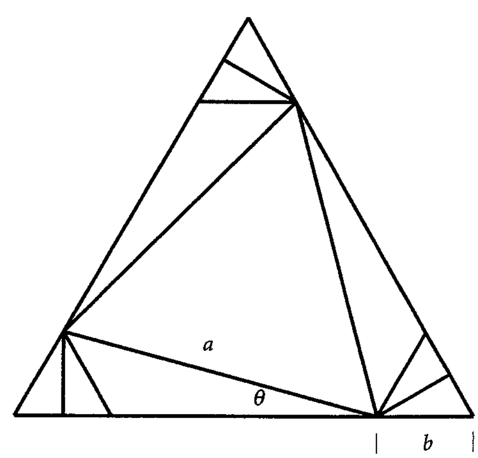

Figure 1

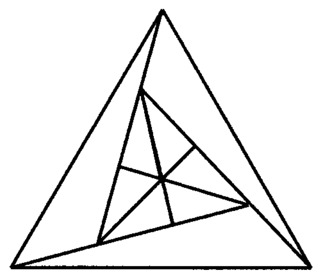

Figure 2

and assembled into the other triangle of side length a, as seen in Figure 2.

Mar 2. John Sampson enjoys Pythagorean triples (positive integers a, b, and c satisfying $a^2 + b^2 = c^2$). He wants you to show that every such triple has one number divisible by 3, one number divisible by 4, and one number divisible by 5. For example in the triple (5, 12, 13) 12 is divisible by 3, 12 is divisible by 4, and 5 is divisible by 5. Show that every Pythagorean triple contains a number divisible by 3, a number divisible by 4, and a number divisible by 5.

Let a, b, and c be a Pythagorean triple. All variables below are positive integers.

One number is divisible by 4: At least one of a, b, and c must be even: since $a^2 + b^2 = c^2$, even if both a and b are odd, c^2, and therefore c, must be even.

Case 1: c is even. Then $c = 2d$, so $a^2 + b^2 = 4d^2$. If a is odd, b is also odd, so $a = 2m-1$ and $b = 2n-1$. Then $(2m-1)^2 + (2n-1)^2 = 4m^2 - 4m + 1 + 4n^2 - 4n + 1 = 4d^2$. Dividing both sides by 4 we get $m^2 - m + n^2 - n + 1/2 = d^2$. However, the left-hand side of this equation is not an integer. Thus, both a and b must be even. Then a and b have the forms $a = 2m$ and $b = 2n$, so $4m^2 + 4n^2 = 4d^2$. Thus $m^2 + n^2 = d^2$. But this is another Pythagorean triple, so one of m, n, or d must be even. Thus, 4 divides a, b, or c.

Case 2: c is odd. Assume that a is odd and b is even. Then $a = 2m-1$, $b = 2p$, and $c = 2n-1$. So $(2m-1)^2 + 4p^2 = (2n-1)^2$, and thus $4p^2 = (2n-1)^2 - (2m-1)^2 = 4n(n-1) - 4m(m-1)$. Dividing both sides by 4, we get $p^2 = n(n-1) - m(m-1)$. The righthand side of this equation is even, so p is also even. Thus 4 divides b.

One number is divisible by 3: If 3 divides a or c, there is nothing to prove, so assume otherwise. Then $b^2 = c^2 - a^2 = (c-a)(c+a)$. Write c and a as $c = 3k+r$ and $a = 3m+p$, with $k, m >= 0$, and $3 > r, p > 0$. Then $c - a = 3(k-m) + (r-p)$, and $c + a = 3(k+m) + (r+p)$. Both r and p are either 1 or 2. If $r = p$, then 3 divides $r - p$. If r and p are distinct, then 3 divides $r + p$. In either case, 3 divides both $3(k-m)$ and $3(k+m)$. Thus, 3 divides either $c - a$ or $c + a$, so 3 divides both b^2 and b.

One number is divisible by 5: If 5 divides a or b, there is nothing to prove, so assume otherwise. The last digits of a and b (base 10) must therefore be 1, 2, 3, 4, 6, 7, 8, or 9. The last digits of a^2 and b^2 are either 1, 4, 6, or 9. The possible last digits of $a^2 + b^2$ are then 0, 2, 3, 5, 7, or 8. Of these, only 0 and 5 are the last digits of any perfect square. Thus, 5 divides both c^2 and c.

BETTER LATE THAN NEVER

1989 F/M 2. This one is "better very late than never." Thomas Jabine writes that his father showed him a "Puzzle Corner" problem from many years ago in which, for each positive integer n, we defined f(n) as the number of times the numeral one is used in counting up to n. So f(1) = f(9) = 1, f(10) = 2, f(11) = 4. We asked for the largest n with f(n) = n, and several readers found the answer 1,111,111,110. Jabine has extended this to number bases other than 10 and obtained

Base	Highest value where f(n)=n.
2	10
3	110
4	1110
5	11110
6	111110
7	1111110
8	11111110
9	111111110
10	1111111110

2000 M/A 3. D. Thomas Terwilliger found a much smaller value.

OTHER RESPONDERS

Responses have also been received from T. Barrows, R. Baum, G. Coram, M. Fisher, M. Fountain, R. Hess, T. P. Jabine, M. Lindenberg, J. Mahoney, L. Nissim, A. Ornstein, E. W. Sard, J. Shapiro, D. Sherman, A. Sledge, D. Smith, M. Teague, J. Tuzson, B. Wake and Q. Watkins.

PROPOSER'S SOLUTION TO SPEED PROBLEM:

Eskimo Pi
1 I.V. League
Won Ton
1 microscope
Mach Turtle
Knot-furlong
1 Rod Serling
1 megahertz

Send problems, solutions and comments to Allan Gottlieb, New York University, 715 Broadway, 10th floor, New York NY 10012, or to gottlieb@nyu.edu.

—Edited by Owen W. Ozier '98