INTRODUCTION

Due to the shift from 8 issues per year to 6, we decided to print solutions two (rather than three) issues after the problems appear. This reduction of one pipeline stage causes a collision, which we are now experience as both the Nov/Dec and Jan/Feb problems are now due for solution. This puts a very serious space squeeze on the column. As a result we are not printing any new problems, which will cause a bubble in the pipeline come Sep/Oct.

SOLUTIONS

1997 N/D 1. Matthew Fountain wants you to find a quadrilateral with sides, diagonals and area all different integers.

A few readers proposed rectangular solutions, which by necessity do not meet the requirement that the diagonals be *different* integers. The solution below is from E. Sard.

Please place figure number 1 here.

Although there may be more general solutions, assume the quadrilateral is inscribable in a circle. The problem is then similar to 1997 M/J 2 for a quadrilateral instead of a pentagon, since, if the sides and radius are different integers, the diagonals and area generally will be too. Referring to the figure, the sides are a, b, c, and d with corresponding central angles 2A, 2B, 2C, and 2D. The diagonals are e and f, the radius is r, and, for clarity, only one perpendicular from the circle center to side a is shown. For each solution, there are six different permutations of side order possible. Also, the circle center will lie outside the quadrilateral if any of the half-angles A, B, C, or D > 90°. The necessary relations are: a=2r sinA, b=2r sinB, c=2r sinC, d=2r sinD, e=2r sin(C+D), f=2r sin(B+C), A+B+C+D=180°, and area=(a/2)r cosA+(b/2)r cosB + (c/2)r cosC + (d/2)r cosD. Note that, the area formula works for D > 90°.

These relations show that the integer requirement implies rational sinA, sinB, sinC, sinD, cosA, cosB, cosC, and cosD with integer r canceling the demoninators. In turn, these rational sines and cosines, the unequal lengths, and an assumed desire for lengths with no common factor imply choosing angles A, B, and C to be the acute angles of right triangles whose sides are primitive Pythagorean triples (PPTs). There are an infinite number of PPTs, the three lowest being 3, 4, 5; 5, 12, 13; and 7, 24, 25. Thus there are an infinite number of quadrilateral solutions. Also, to ensure D > 0°, A+B+C < 180°, and D (which can be obtuse) is calculated from cosD=-cos(A+B+C). This relation also works for D > 90° and reduces to cosD=sinC or sinA for A+B=C+D=90° (e a diameter) or B+C=90° (f a diameter), respectively.

Two examples are given, the first with A and B from one right triangle (and thus, C and D are from a second right triangle), and the second with A, B, and C from three different right triangles. Let sinA=3/5, sinB=4/5, and sinC=5/13=cosD. The complementary values are cosA=4/5, cosB=3/5, and cosC=sinD=12/13. Expanding, sin (B+C)=63/65. Choose r=(5)(13)/2 to give a=39, b=52, c=25, d=60, e=65, f=63, and area=1764. Let sinA=3/5, sinB=5/13, and sinC=24/25. Then, expanding, cosD=1113/1625. The complentary values are cosA=4/5, cosB=12/13, cosC=7/25, and sinD=1184/1625. Expanding, sin(B+C)=323/325, and sin(C+D)=56/65. (Note, B+C and C+D > 90°). Choose r=(25)(65)/2 to give a=975, b=625, c=1560, d=1184, e=1400, f=1615, and area=1058148.

N/D 2. Richard Hess, a veteran of 25+ years of ‘‘Puzzle Corner’’ wants you to show that

$$\cos(2\pi/17)\cdot\cos(4\pi/17)\cdot\cos(6\pi/17)\cdot\cdots\cdot\cos(16\pi/17) \;=\; 1/256$$

Henri Hodara was able to prove a more general result: For any odd N

$$\prod_{m=1}^{(N-1)/2}\cos\left(\frac{2\pi m}{N}\right) = \frac{\cos(((N-1)/2)(\pi/2)) - \sin(((N-1)/2)(\pi/2))}{2^{(N-1)/2}}$$

The proposer, using Tchebychev polynomials, found a short proof of the stated problem. Matthew Fountain found the following cute proof using the double angle formula.

By substitution of $\cos(x) = \sin(2x)/2\sin(x)$,

$$\cos(2x)\cos(4x)\cos(6x)\cdots\cos(16x) = \frac{1}{256}\,\frac{\sin(4x)\sin(8x)\sin(12x)\cdots\sin(32x)}{\sin(2x)\sin(4x)\sin(6x)\cdots\sin(16x)}$$

$$= \frac{1}{256}\,\frac{\sin(20x)\sin(24x)\sin(28x)\sin(32x)}{\sin(2x)\sin(6x)\sin(10x)\sin(14x)}\,.$$

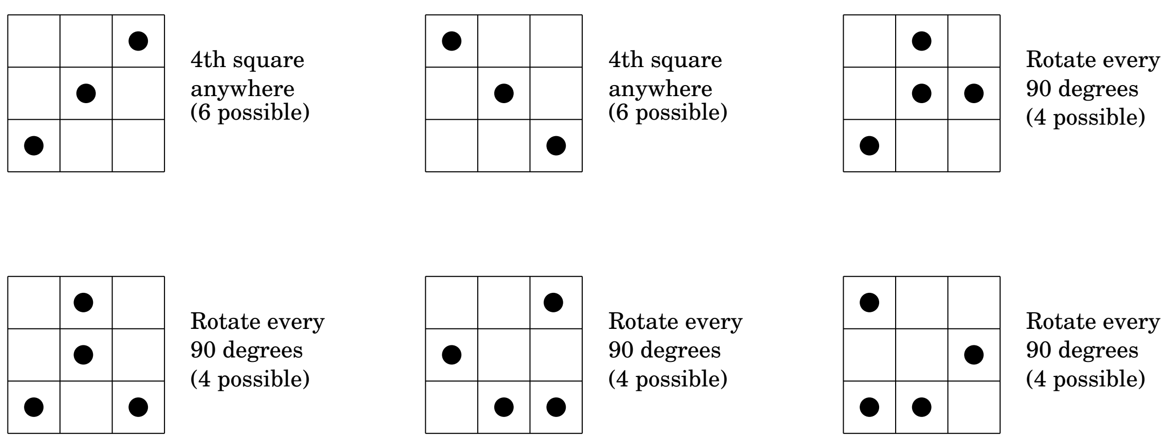
When $x = \pi/17$, $\sin(20x) = -\sin(14x)$, $\sin(24x) = -\sin(10x)$, $\sin(28x) = -\sin(6x)$, $\sin(32x) = -\sin(2x)$, and the sine terms in the fraction cancel out leaving 1/256.

N/D 3. We end with a problem from Kelly Woods. Consider nine squares in a three-by-three arrangement as in tic-tac-toe. If three different squares are selected at random, what is the probability that they will be in a straight line (horizontal, vertical, or diagonal)? If four different squares are selected at random, what is the probability that three of them will lie in a straight line? Similarly for selecting five and six squares.

Ken Rosatto writes: For n=3, the number of possible arrangements is $\binom{9}{3} = 84$. There are only 8 ways that 3 squares can line up in a row (3 across, 3 down, 2 diagonal). So $P_{(n=3)} = 8/84 = 2/21$.

For n=4, the number of possible arrangements is $\binom{9}{4} = 126$. For each of the 8 ways above (for n=3) there are 6 possible places to put the 4th square. So $P_{(n=4)} = (8 \cdot 6)/126 = 8/21$.

For n=5, the number of possible arrangements is $\binom{9}{5} = 126$. The easiest way to find the number of arrangement having 3 in a row (since with 5 squares there can be two 3-squares-in-a-row arrangements) is to find the number of 4 square arrangements that block all 3-in-a-row arrangements. They are



So $P_{(n=5)} = \dfrac{126 - (6+6+4+4+4+4)}{126} = \dfrac{7}{9}$.

For $n = 6$, the number of possible arrangements is $\binom{9}{6} = 84$. The only two that do not work are missing the three squares in a diagonal (i.e. the first two diagrams above). So $P_{(n=6)} = (84-2)/84 = 41/42$.

1998 J/F 1. Charles Wampler has a cube of marble, measuring 1 meter on a side. For an element in his newest modern masterpiece, he would like to carve from the cube a tetrahedron having the largest possible volume. How long are the edges of the tetrahedron and what is its volume? Since the grain of the stone is not uniform, the sculptor would also like to know if there is any choice in how the tetrahedron is positioned within the cube.

Ronald Ouellette tells us that the edges of the tetrahedron are of length $\sqrt{2}$ meters, and its volume is 1/3 cubic meters. There are two choices for the tetrahedron inside the cube. Ouellette notes that the largest tetrahedron that can be fit inside the cube has all six of its edges as diagonals of the six faces of the cube, as shown in the figure below from Tim Barrows. Hence the edge length is $\sqrt{2}$ meters. Ouellete observes that the volume in cubic meters of one of the corner pieces removed is

$$\frac{1}{6} = \int_0^1 \frac{(1-x)^2}{2}\,dx$$

Hence the volume of the tetrahedron is $1 - 4(1/6) = 1/3$ cubic meters. Barrows is able to obtain this volume without calculus, but the calculation is longer. Choosing one of the two diagonals for any face determines the tetrahedron so there are exactly two possibilities.

Please place figure number 2 here.

J/F 2. Here is one Norman Spencer found in *Better Homes and Gardens*. How can you divide a circle into n equal sized segments? Note that by a circle we mean the one dimensional object sometimes (in the editor's view, erroneously) called the circumference of the circle. In addition to the using the customary compass and straight-edge, you may divide a *line* into n equal size pieces.

I must have been somewhat asleep at the switch. As was pointed out by several readers (more politely than perhaps I deserved), it is well known that there are many n for which one cannot construct an angle of $360/n$ degrees. Hence there can be no exact solution to this problem and the method in *BH&G* must be an approximation.

BETTER LATE THAN NEVER

1997 Jul 1. Max Gellert believes that a club lead defeats the contract.

Jul 2. William Peirce found explicit expressions for a and b.

OTHER RESPONDERS

Responses have also been received from H. Amster, Auran, D. Cohen, K. Duisenberg, R. Giovanniello, F. Grosselfinger, J. Grossman, H. Hodara, S. Morgan, C. Muehe, A. Mullin, R. Sinclair, and C. Wiegert.