INTRODUCTION

As you may recall, during 1996-97 I took an unpaid leave from NYU to spend a year at the NEC Research Institute in Princeton NJ. I have returned to NYU but did enjoy my stay at NECI and will continue there this year in a consulting status. Officially, I am a Professor of Computer Science at NYU and an Acting Fellow at NECI. Mail, both hard copy and electronic, can be sent to me at either location.

I write this column on 10 September having recently given one 2-hour lecture in each of the two courses I teach this semester (graduate operating systems on Tuesday evenings and graduate computer systems design on Thursday evenings). I hadn't taught for a year and forgot just how much I enjoy it. I also forgot how physically tiring it is. I guess the adrenaline flows during class as I felt exceptionally lively and the students commented on my energy level. But on the train home at 8PM, I had to concentrate to stay awake. I wonder if this is just another sign of my being ‘‘experience advantaged’’ (a marvelous PC term I learned from NYU President L. Jay Oliva).

PROBLEMS

N/D 1. Matthew Fountain wants you to find a quadrilateral with sides, diagonals and area all different integers.

N/D 2. Richard Hess, a veteran of 25+ years of ''Puzzle Corner'' wants you to show that

$$\cos(2\pi/17) \cdot \cos(4\pi/17) \cdot \cos(6\pi/17) \cdot \cdots \cdot \cos(16\pi/17) = 1/256$$

N/D 3. We end with a problem from Kelly Woods. Consider nine squares in a three-by-three arrangement as in tic-tac-toe. If three different squares are selected at random, what is the probability that they will be in a straight line (horizontal, vertical, or diagonal)? If four different squares are selected at random, what is the probability that three of them will lie in a straight line? Similarly for selecting five and six squares.

SPEED DEPARTMENT

Here is one sent in by former *Technology Review* editor, John Mattill. The problem was created by Alan Guth for an inter-class competition at a recent MIT reunion. Suppose there is a planet twice as far from the sun as Earth is. How many Earth years would it take for the planet to revolve around the sun?

SOLUTIONS

Jul 1. We begin with a Bridge problem from Leonard Nissim that he believes came from *Bridge World* magazine in the 1980's.

You are South, and will be the declarer in an undoubled contract. West will lead the King of Spades against any contract, and you are to presume best defense after that lead. What contract would you like to be in?

North
S 7 5 3 2
H A Q
D A K Q T
C K J 9

West
S K Q J
H K J
D J 9 8 7
C Q T 8 7

East
S T 9
H T 9 8 7
D 6 5 4 3
C 5 4 3

South
S A 8 6 4
H 6 5 4 3 2
D 2
C A 6 2

The following solution is from Bruce Layton.

The contract I want to be in is seven clubs. West's spades allow him to defeat a spade slam; ditto East's hearts a heart slam. If both of them hang onto those suits, they can defeat a slam in NT or diamonds. The way to make 7 clubs follows; note that West must follow the first nine tricks, and East the first eight; if East had the C6, and South the C5, East could defeat 7C by playing the C6 at trick nine, a true milestone in the annals of uppercutting. Win the spade lead with the ace in hand. Finesse the D 10. Cash the three top Ds, discarding your three Spade losers. Ruff a spade with the C2. Finesse the HQ. Cash the HA. Ruff another spade with the C6, as East curses his luck in having nothing higher. Now, West has nothing left but trump, so he must ruff your heart lead. Overruff as economically as possible. Ruff the last spade in hand with the CA. For the last two tricks, you have the highest and third-highest trumps on the board, and West has the second-highest, so you don't need to have a trump left to finesse him.

Jul 2. Tom Harriman wants to know the positive integer solutions to

$$2a^2 = b^2 + 1$$

Charles Rife notes that there are an infinite number of solutions and that the problem is solved by recourse to the Pell equation. He further remarks that he learned about the Pell equation while studying a problem from ‘‘Puzzle Corner’’ roughly 25 years ago! Matthew Fountain raves about the CD version of the Encyclopedia Britannica and notes that Pell’s equation is well described there. Fountain, however, solved Lehman’s problem *before* finding the material in EB. Jeff Dike sent us the following solution.

The partial fraction expansion of $\sqrt{2}$ is [1 2 2 2 ...]. Every other convergent (b/a), starting with 1/1, satisfies $2a^2 = b^2 + 1$. The first few are 1/1, 7/5, and 41/29. The other convergents satisfy $2a^2 = b^2 - 1$. I will prove by induction that every other convergent satisfies the equation. The base case is 1/1, which can be observed to satisfy the equation. Each convergent, in terms of the previous convergent, is $(2a + b)/(a + b)$. Iterating this twice to get each convergent in terms of the second previous convergent yields $(4a + 3b)/(3a + 2b)$. We have $2a^2 - b^2 = 1$. Expanding $2(3a + 2b)^2 - (4a + 3b)^2$ and simplifying gives $2a^2 - b^2$, which equals 1, and establishes the induction.

To demonstrate that these are all of the solutions, I will use the fact that if a rational number b/a differs from an irrational number by less than $1/2a^2$, then b/a is one of the convergents of the irrational number. If all solutions of $2a^2 = b^2 - 1$ differ from $\sqrt{2}$ by less than $1/2a^2$, then they are all convergents of $\sqrt{2}$ and included in the solution given above.

To calculate $\left| b/a - \sqrt{2} \right|$, rearrange $2a^2 = b^2 - 1$ to $b^2/2a^2 = 1 + 1/2a^2$ Taking the square root and subtracting one from both sides gives

$$(1/\sqrt{2})(b/a - \sqrt{2}) = \sqrt{1 + 1/2a^2} - 1$$

Multiplying by $\sqrt{2}$ gives

$$(b/a - \sqrt{2}) = \sqrt{2}(\sqrt{1 + 1/2a^2} - 1) = E$$

Expanding $\sqrt{1 + 1/2a^2}$ around 0 to two Taylor series terms and a remainder gives $1 + 1/4a^2 + R$, where $R = (f''(ta)a^2)/2, \text{with} 0 < t < 1$. Calculating R gives $-((1 + ta)^{(-3/2)}a^2)/8$, which is negative, and thus bounded above by 0. Substituting $1 + 1/4a^2$ for $\sqrt{1 + 1/2a^2}$ in the value of E above gives $\sqrt{2}(1 + 1/4a^2 - 1) > (b/a - \sqrt{2})$. Simplifying this gives $1/(2\sqrt{2}a^2) > b/a - \sqrt{2}$. To conclude this argument, I observe that $1/2a^2 > 1/(2\sqrt{2}a^2) > b/a - \sqrt{2}$.

Jul 3. This nautical challenge is due to Eric Lehman.

Suppose you are floating in a sea of 7's, and you have a raft with the number 101. You discover that you can take a 7 and insert it into your raft to enlarge the number (getting 7101, 1701, 1071, or 1017). Unfortunately, every time you do this, the number divides itself by its smallest prime factor (leaving, from the above example, 2367, 567, 357, or 339). You know that if the number goes below 100, your raft will sink. What is the longest you can survive?

Reid Simmons found that a little C was helpful in staying afloat in a big sea (of sevens). I have also heard that you need to roll a bunch of 7's to stay afloat in C, but that is another story. Simmons writes. Fortunately for the clever mathematician (or the clever computer scientist with access to a hand-held computer and the ability to write a short C program), he can survive indefinitely in the raft floating in a sea of 7's.

There may be additional cycles, but the one that I (well, my program) found is:

```
101 =>
 1017/3 = 339 =>
  7339/41 = 179 =>
   1797/3 = 599 =>
    5997/3 = 1999 =>
     17999/41 = 439 =>
**    4379/29 = 151 =>
       1751/17 = 103 =>
        1703/13 = 131 =>
         1317/3 = 439 => **
```

OTHER RESPONDERS

Responses have also been received from S. Brown, W. DeHart, P. Flanagan, A. Francoeur, R. Giovanniello, D. Greenberg, J. Harmse, J. Horton, R. Mallett, C. Muehe, B. Norton, A. Ornstein, G. Perry, K. Rosato, F. Rubin, E. Sard, D. Savage, and J. Stuart.

PROPOSER'S SOLUTION TO SPEED PROBLEM

$2\sqrt{2}$ years.