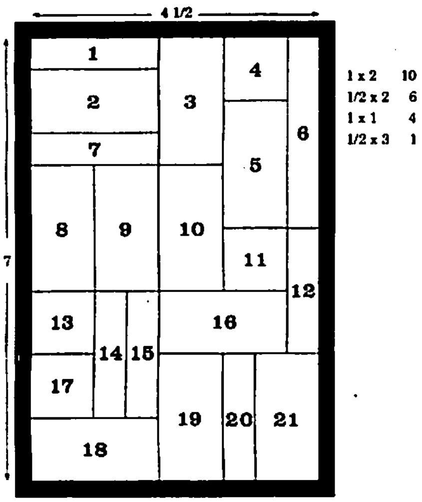
Tackling the Blocks

It has been a year since I specified the size of the backlogs for the various kinds of problems that are printed; let me do so now. I have about a two-year supply of regular problems, and close to a year of chess, bridge, and speed problems; computer problems are in short supply.

Problems

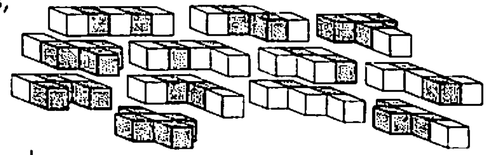
APR 1. Our first offering is a chess/timber problem from Winslow Hartford, who begins by pointing out that a closed knight's tour is important for timber companies since it gives a pattern for cutting sections of a forest so that successive cuts are not on adjacent land. Even better would be a closed tour of a "superknight" that moves 3 squares in one direction and 2 in the other (instead of the usual 2 and 1). The smallest chessboard containing a closed knight's tour is a 6x6. What is the smallest chess board containing a closed superknight's tour?

APR 2. Samuel Gluss has a set of blocks that fits nicely into a wooden box as shown below. His father, David, notes that there are many ways to pack the pieces into the box and wants you to find one with the 1/2x3 piece (#6) placed horizontally in the upper left corner.



SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012, OR TO: GOTTLIEB@NYU.EDU

APR 3. Nob Yoshigahara wants you to pile up the 12 pentacubes shown into a 3x4x5 solid. Nob adds that the solution is unique up to mirror image.



Speed Department

Phil Bonomo offers one for the "space cadets." Circular orbital speed for the synchronous (24-hour period) class of communications satellites is about 10,000 fps. Without consulting a table of physical constants, what is the circular orbital speed for the sub-synchronous (12-hour period) class of navigation satellites?

Solutions

N/D 1. We begin with a computer-related problem that Max Hailperin heard from Albert Faessler. A primitive Pythagorean triple (PPT) is a triple of positive integers (a,b,c) such that $a^2+b^2=c^2$ and a, b, and c have no common factor (this last condition is what makes the triple "primitive"). The area of a PPT is ab/2. Euler found that the smallest area shared by three PPTs is 13123110. What is the next smallest area shared by three PPTs?

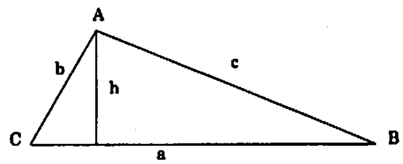
I suspect this problem is far from trivial. Only the proposer sent a solution, namely 2203385574390, which is shared by the three PPTs: (376420, 11707059,11713109), (403332,10925915,10933357), and (1082620,4070469,4211981). He has sent a copy of his program. Due to space considerations we are printing only the introductory comment, which gives the method. Readers wishing the actual code should send a letter to Faith Hruby at *Technology Review*.

It has been known at least since Euclid that the PPTs (x, y, z) described above are in one-to-one correspondence with the "generator" pairs (a,b) with gcd(a,b)=1, (a+b)(mod 2)=1, and a>b. (Let $x = 2ab$, $y = a^2-b^2$, $z = a^2+b^2$.) The area of a PPT in terms of its generator pair is f(a,b) = ab(a-b)(a+b).

The approach taken is to enumerate pairs of numbers in order of increasing f-value, with an eye out for consecutive sequences of pairs that have equal f-values. The way the pairs are enumerated in increasing f-value order is by using a change of variables d=a-b to re-express the problem in terms of f(b,d) = bd(b+d)(2b+d) which is monotonic in both b and d. (The restrictions are now that d must be odd and b and d relatively prime.) Since this rewritten f is monotonic in b and d, we know that at any given point in the enumeration, the next pair must either be an already used b-value together with the next-higher d-value than it was used with, or (bcount+1, 1) where bcount is the number of distinct b-values which have been used so far in the enumeration. A heap priority queue is used to keep track of which of these various possibilities is in fact next in the enumeration order, and Euclid's algorithm for the greatest common denominator is used to test for relative primality. The f-values are represented as double-precision floating point numbers, rather than unsigned integers, because 32 bits isn't enough, but 53 is.

N/D 2. Gordon Rice wants you to find (non-equilateral) triangles containing a 60° angle. How about a 30° angle?

The following solution is from Matthew Fountain: The triangle with sides a=8, b=3, and c=7 has a 60° angle between a and b. No triangle with integer sides contains a 30° angle. Let a and b be the sides of a triangle ABC with C=60° and a>b. Altitude h, perpendicular to a, is of length $(b)\mathrm{Sin}(60°)=(b)(3/4)^{(1/2)}$ and meets side a at distance b/2 from C. The area enclosed by h, c, and part of a is a right triangle with sides of length h, c, and (a-b/2). Let d=a-b/2. Then $c^2-d^2=h^2=(3/4)b^2$. Setting X=c-d, and solving we obtain $c=3b^2/8X+X/2$, and $d=3b^2/8X-X/2$. We now can select values for b and X that produce an integer c. Then a=d+b/2 is also an integer. For example, when X=2, b=4 produces c=4 and a=4, b=8 produces c=13 and a=15, b=12 produces c=28 and a=32. The formula $c^2=a^2+b^2-(2ab)\mathrm{Cos}(C)$ may be used to check these values. The same formula shows that c cannot be an integer when $\mathrm{Cos}(C)=(3/4)^{(1/2)}$.



N/D 3. Tom Harriman wonders for what values of X does the fol lowing "infinite exponential" converge.

$$X^{X^{X^{\cdot^{\cdot}}}}$$

Tricky! X=-1 clearly works and $X>e^{1/e}$ diverges but the exact answer is not so easy. X=-1 is the only negative answer. The non-negative solutions are

$$(1/e)^e \le X \le e^{(1/e)}$$

In particular, $0<X<(1/e)^e$ does not work. Instead, as noted by G. Blondin, for such X, the exponential converges to oscillation between A and B, where $X^A=B$ and $X^B=A$, with

$$X<A<1/e<B<1 .$$

Better Late Than Never

Jul 2. Oops!! The solution method given in Nov/Dec guarantees a loss, not a win. Below is a correct solution from Ed Sheldon, who remarks that overlooking this error must have been due to my close encounter with Hurricane Andrew. The problem was as follows:

Matthew Fountain suggests we tackle the "hold that line" problem devised by Sid Sackson and appearing in his book *A Gamet of Games*.

"Hold That Line" is a game in which two players alternate drawing straight lines between dots on a 4 x 4 dot field. The player to draw the last line loses. The first diagram shows a game in which the lines are numbered in the order they were drawn. Restrictions are that lines after the first shall only be

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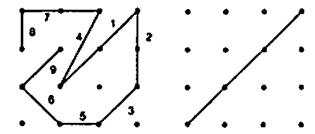
drawn from the free end of a previously drawn line. All lines must be straight and start and end at a dot. A line may connect more than two dots if all are in a straight line. No line shall be drawn to a previously connected dot or cross another line.

The second diagram shows a game in progress where the first player has drawn his first line along a long diagonal of the field. Is this a winning or a losing move? There can be no ties.

Sheldon asserts that the long diagonal is a winning move and writes:



GAME LOST BY FIRST PLAYER   GAME IN PROGRESS

Consider the points in the matrix numbered (in hexadecimal notation) as follows:

```
C D E F
      /
8 9 A B
    /
4 5 6 7
  /
0 1 2 3
```

After White's first move, assume Black plays from 0 to a point in the lower right half. For every Black play, except 0>B, White has a winning play into the same half. Therefore Black's first move is forced: 0>B. The half-court winning plays are as follows:

```
White : Black : White : Black : White : Black : White : Black:
0 > F : 0 > 1 : F > 2 : 1 > 6 : 2 > 7 : 7 > 3 : Wins
                      :       :       : 7 > B : Wins
                      : 2 > 3 : 3 > B : 1 > 6 : Wins
                      : 2 > 6 : Wins
                      : 2 > 7 : 7 > 3 : 1 > 6 : Wins
                      : 2 > B : B > 3 : 1 > 6 : Wins
0 > F : 0 > 2 : F > 3 : * > 6 : Wins   ( * = either way)
0 > F : 0 > 3 : F > 7 : * > 6 : Wins
0 > F : 0 > 6 : F > 7 : 6 > 1 : 1 > 2 : * > 3 : Wins
                      : 6 > 2 : 7 > 3 : 2 > 1 : Wins
                      : 6 > 3 : 3 > 2 : 2 > 1 : Wins
                      : 7 > * : Wins (mirror of above 6 moves)
0 > F : 0 > 7 : F > B : B > 6 : 7 > 2 : 2 > * : Wins
                      : 7 > 1 : 1 > 3 : B > 6 : Wins
                      : 7 > 2 : 2 > 1 : B > 6 : Wins
                      : 7 > 3 : 3 > 1 : B > 6 : Wins
0 > F : 0 > B : Forced move for Black.
```

Black's forced move creates two independent half-court games. In line with the old NIM philosophy of matching moves until the end we will assume that white makes a symmetrical game by playing F>4, which since the games are independent can be treated as a 0>B move, and for convenience, both will be treated as 0>1 moves. The resulting 5 point games will start:

```
Five Point Game:          1 > 2 > 3 > 6 > * > * :
                          1 > 2 > 3 > 7 (mirror of 1 2 3 6)
                          1 > 2 > 3 > B > 6 :
                          1 > 2 > 6 > 3 > 7 > B :
                          1 > 2 > 6 > 3 > B :
                          1 > 2 > 6 > 7 > * :
                          1 > 2 > 6 > B (mirror of 1 2 6 3)
                          1 > 2 > 7 > 3 :
                          1 > 2 > 7 > 6 > B :
                          1 > 2 > 7 > B (mirror of 1 2 7 6)
                          1 > 2 > B > 3 :
                          1 > 2 > B > 6 :
                          1 > 2 > B > 7 > 3 :
                          1 > 3 > 6 > * > * :
                          1 > 3 > 7 (mirror of 1 3 6)
                          1 > 3 > B > 6 :
                          1 > 6 (mirror of 1 2-->)
                          1 > 7 > * > * :
                          1 > B (mirror of 1 3-->)
```

Removing mirror image entries, we obtain (the even-odd parity includes 0>1, but not 0>F):

```
1 > 2 > 3 > 6 > * > * : Even
        3 > B > 6 : Odd
        6 > 3 > 7 > B : Even
        6 > 3 > B : Odd
        6 > 7 > * : Odd
        7 > 3 : Even
        7 > 6 > B : Odd
        B > 3 : Even
        B > 6 : Even
        B > 7 > 3 : Odd

1 > 3 > 6 > * > * : Odd
        B > 6 : Even
1 > 7 > * > * : Even
```

For a White win, one "game" must be played even and the other odd.

If Black plays 1>3 in one half, White plays 1>3 in the other. Black then must choose 6 or B (Odd or Even) in one half, and White plays opposite for Even/Odd and a win.

If Black plays 1>7 or 1>2 in one half, White plays 1>2 or 1>7 in the other. The Even game is a wash, and the situation reduces to a single game with Black selecting 3,6,7 or B from the following board:

```
1 > 2 > 3 > 6 > * > * : Even
            B > 6 : Odd
        6 > 3 > 7 > B : Even
            3 > B : Odd
            7 > * : Odd
        7 > 3 : Even
            6 > B : Odd
        B > 3 : Even
            6 : Even
            7 > 3 : Odd

If Black 2 > 3, White 3 > B Wins.
If Black 2 > 6, White 6 > 7 Wins.
If Black 2 > 7, White 7 > 6 Wins.
If Black 2 > B, White B > 7 Wins.
```

The above shows that White has a forced win for all legal moves by Black. Therefore the answer is that playing the major diagonal is a winning first move.

Other Responders: Responses have also been received from M. Astelfi, R. Hess, E. Kaplan, J. Keilin, A. Mullin, J. Preston, E. Sard, S. Shaffer, T. Shepard, R. Stanley, K. Turkewitz, and D. Ullery.

Proposer's Solution to Speed Problem: 12,600 fps.