ALLAN J. GOTTLIEB, '67

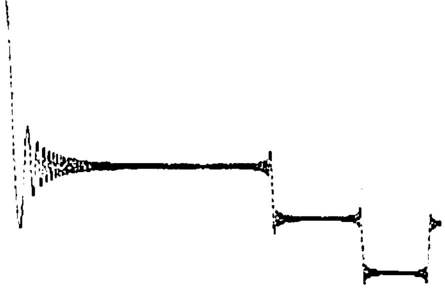
Milligram Cracker

It has been at least a year since I specified the size of the backlogs for the various kinds of problems that are printed. Let me do so now. We have about a year's supply of bridge, speed, and regular problems; about a half-year's backlog of computer-related problems; and as of today NO chess problems. Speedy Jim Landau (who just completed a master's degree in computer engineering; congratulations, Jim) reminds us that computer problems need not be straight programming assignments. For one example, see his **APR 1** below; for others consider computer architecture, finding bugs (hopefully cute ones) and figuring out what uncommented problems actually do.

I owe David Wagger an apology. He submitted two problems to *GSC NEWS*, a crossword puzzle and a cryptoquiz. The crossword puzzle was misprinted by *GSC NEWS* and the same incorrect version was reprinted in "Puzzle Corner". Even worse, the cryptoquiz, which was printed correctly in *GSC NEWS*, was mangled when reprinted in "Puzzle Corner" (a word was omitted and Wagger was misspelled).

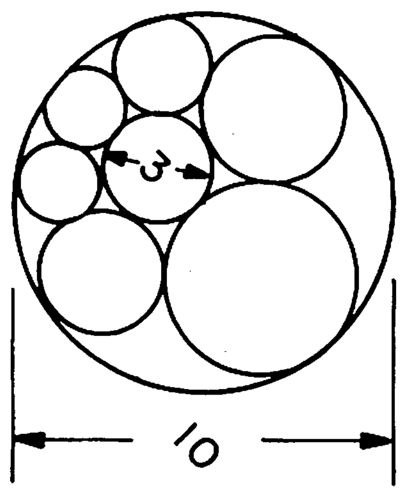
Problems

APR 1. We begin with a computer-oriented problem from Jim Landau, who wants you to write a program that generates (a reasonable facsimile of) the following graphic output.

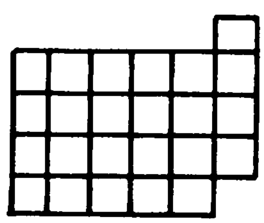


APR 2. Frank Rubin wants the largest prime having a digit that can be replaced by any of the nine other digits with the resulting number still prime.

APR 3. Matthew Fountain knows only that the inner and outer circles in the figure below have diameters of 3 and 10 but still wants you to determine the distance between their centers.



APR 4. The following problem appeared in "Golomb's Gambits" edited by Solomon Golomb in the *Johns Hopkins Magazine*. You are to divide the figure below into four congruent pieces. There are two solutions.



APR 5. Robert Bart's hypertension medication comes in 5 mg. tablets that can be divided in half to give morning and evening doses of 2.5 mg. each. At each dose, Bart selects a pill at random from the bottle. If it is 2.5 mg., it is used; if it is 5 mg., half is used and half returned to the bottle. Ignore size effects, i.e. the chance of getting a large pill is simply the percentage of pills that are large. Originally, Bart has N 5 mg. tablets and he wants to know what is likely to be the number of 5 mg. tablets remaining after i days. More specifically, he wants you to determine $P(n,N,i)$, the probability that n 5 mg. tablets remain after i days of use starting from N 5 mg. tablets and then determine the expected value

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

of n as a function of N and i, i.e.,

$$E_n(N,i) = \sum_1 nP(n,N,i)$$

In particular, what is $P(1,N,N-1)$, the probability that the *last* two doses will be a single 5 mg. tablet rather than two 2.5 mg. pieces? If a closed form is not possible, how about numerical solutions for $P(50,100,50)$, $E_n(50,20)$, $E_n(50,40)$, $P(10,20,10)$, and $E_n(10,5)$.

Speed Department

SD 1. Mark Astolfi wants you to imagine that the National Football League abolished the 2-point safety. What scores (for one team) would become impossible?

SD 2. Eric Weill notes that there are 13 diamond *cards* in a card deck. How many diamonds are on those 13 cards?

Solutions

N/D 1. Matthew Fountain proposes, as a computer-oriented problem, finding the smallest nonprime integer, $N > 1$, that divides $2^{N-1} - 1$. As a noncomputer addition, explain why there are so "few" such Ns.

We give two solutions below, from Bob High and Howard Helman. Samuel Wagstaff, Jr., sent us a paper he cowrote on this subject, copies of which are available from the editor. Two interesting points found in that paper are that there are 14,884 (odd) solutions below 10^{10} and a reference to a result of Erdos that there are no more than

$$x\exp(-c\sqrt{\ln x \ln\ln x})$$

such solutions below x.

Helman writes:

My first thought to solve the problem was to use a programming language which has infinite precision arithmetic. I had developed such a language modeled after TRAC and SAM76. After analyzing the problem, I noticed that it required two critical functions: 1) checking if a number is prime and 2) checking if a number evenly divides a number that in binary consists of a string of ones. If the solution is less than the size of a word on the computer, the prime check requires no multiple precision arithmetic and the multiple precision divide check can be simply done because the dividend consists of all ones (actually (n-1) of them) and we need only to check remainders. The top-down approach to solving the problem is as follows:

The main routine calls the check routine with n starting at 2 and incrementing by one until check returns a true. The result is then printed.

The check routine checks to see if n satisfies the problem constraints. Since 2 raised to any power minus one is odd, all divisors must be odd as well as nonprime. Therefore, check quickly rejects any even or prime n. For the remaining cases a number whose binary representation is (n-1) ones must be checked to see if it is evenly divisible by n. Assuming at least 16 bits in a machine word and a processor that can do 32-bit arithmetic, the process is the one one would use to divide a large number by a single digit divisor. Here we use 16-bit digits. The high order digit has (n-1) mod 16 ones in it and it is followed by (n-1)/16 digits of all ones. To generate the high order digit a shift is used to generate 2 to the x where x is (n-1) mod 16. Then subtract one to get x ones. At each step we only need the remainder (rem) from the previous step. Rem is shifted over 16 bits and the next digit (16 ones) is appended and the remainder is again computed. If the final remainder is zero, n divided our big string of ones, otherwise this n failed.

All that is left is a prime checking routine and prime(i) is a simpleminded one. Checks are made to see if i is even or is one of the primes 2 or 3. Otherwise i is divided by odd numbers less than the square root of i until all have been checked or a zero remainder is found. If no odd evenly divided i, i is prime.

The following C program incorporates the above design and it gave 341 as the first value. The program uses very little memory and yet $2^{340} - 1$ is 103 digits long.

```
#include <stdio.h>          /* C I/O definitions */
/* prime(i) returns 1 if and only if i is prime*/
int prime (int i)
 {int j, result;
  if(i < = 3)    result = 1;
   else if( !i%2) result = 0;
   else {
   j = 3;  .
    while(i%j&&j < i/j)
      j = j + 2;
    result = (i%j)! = 0;
   }
   return result;
 }

/*check(n) returns 1 if and only if n satisfies problem
requirements*/
int check(int n)
 {unsigned long rem,t;
  int i;
  if((n%2) = = 0 || prime(n)) rem = 1;
  else {
   t = (1 < < ((n - 1) % 16)) - 1;
   rem = (t - 1) % n;
   for (rem = t % n, i = (n - 1) / 16; i; i--)
    rem = (( rem < < 16) + 0xffffU) % n;
  }
 .return (rem = = 0);
}

void main(void){int n;
 for (n = 2; !check(n); n + +);
  printf("the first is %d0,n);
}
```

Bob High's solution follows:

The smallest such number is 341. The following is a list of all such numbers under 10,000: 341, 561, 645, 1105, 1387, 1729, 1905, 2047, 2465, 2701, 2821, 3277, 4033, 4369, 4371, 4681, 5461, 6601, 7957, 8321, 8481, 8911.

If we have

$$2^{n-1} \equiv 1 \ mod\ n \qquad [1]$$

let k be the smallest integer such that

$$2^k \equiv 1 \ mod\ n$$

Then clearly we must have $k|n-1$, and k must divide the order of the multiplicative group of integers modulo n as well. This group has order $\phi(n)$, where ϕ is the Euler function, which counts the number of integers less than and relatively prime to n. We have that:

$$\phi(p^k) = (p-1)p^{k-1}$$

for p prime, and

$$\phi(mn) = \phi(m)\phi(n)$$

for m and n relatively prime.

This says that we must in fact have:

$$2^{(n-1,\phi(n))} \equiv 1 \ mod\ n \qquad [2]$$

where $(n-1,\phi(n))$ is the *greatest common divisor* of $n-1$ and $\phi(n)$. Very few integers n are likely to satisfy this relationship.

For 341, for example, we have that $\phi(341) = 30 \times 10 = 300$, and $(n-1,\phi(n)) = (340, 300) = 20$. In fact, $k = 10$, since $2^{10} = 1024 \equiv 1 \ mod\ 341$, and since $10 | 20$, 341 works.

Actually, [1] is a well-known test for primality—or, rather, non-primality, since if n *fails* to meet condition [1] it *must* be composite. Of course, the same test can be performed with other primes p in place of 2:

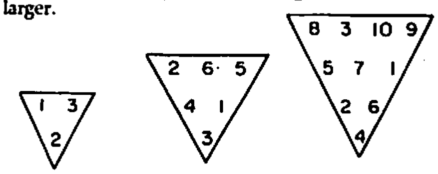
$$p^{n-1} \equiv 1 \ mod\ n \qquad [3]$$

For example, for $p = 3$, we have the following values: 91, 121, 671, 703, 949, 1105, 1541, 1729, 1891, 2465, 2665, 2701, 2821, 3281, 3367, 3751, 4961, 5551, 6601, 7381, 8401, 8911, and for $p = 5$, we have: 217, 561, 781, 1541, 1729, 1891, 2821, 4123, 5461, 5611, 5731, 6601, 7449, 7813, 8029, 8911, 9881.

Note that the number 1729 is included in all three of these lists. This is *another* way in which 1729 is special (1729 is the number Ramanujan is supposed to have remarked upon to someone—Hardy?—on his deathbed, noting that it was "the smallest number that can be expressed as the sum of two cubes in two distinct ways"). Since a prime must satisfy *all* such congruences, this provides a strong (but not infallible!) test for primality.

Also solved by Edwin Kruse, Robert Moeser, George Welti, Winslow Hartford, Richard Tooley, Harry Zaremba, Steven Feldman, Larry Bell, and the proposer.

N/D 2. Nob. Yoshigahara wants you to find the rule for each of the following triangular configurations of numbers and to produce a configuration one size larger.



The following solution is from first-time responder William Kampe:

The first trick is to figure out what the puzzle really requires. A quick look indicates the challenge is to fill a size five triangle with the integers 1, 2, 3, . . . 15 such that

a. The 5 is at the bottom apex.

b. For rows 2 thru 5 (starting at the top) each integer is the absolute difference of the two integers above.

A little thought shows that the 15 must be in the top row. Further, 12, 13, and 14 must be high up, with maybe a couple of the trio in the top row. After a little trial and error, I felt the number of combinations to try were too much for trial and error.

Then I thought about the different patterns of odds and evens that could lead to an odd at the bottom apex, and have the right number of 8 odd and 7 even numbers. A quick test of the possible arrangements of the top row yields only three valid arrangements of odds and evens in the top row (not counting mirror images):

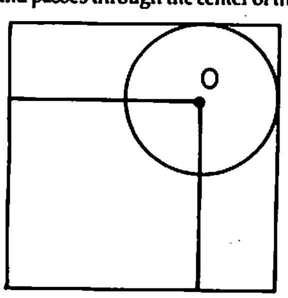
1.	2.	3.
o e o e e	o o o o e	o o o e e
o o o e	e e e o	e e o e
e e o	e e o	e o o
e o	e o	o e
o	o	o

A bit of trial and error here shows that there are actually very few combinations to test for each of the patterns, especially since 15 and the other high numbers must show up in the top two rows. As luck would have it, the solution came from pattern 3:

13 3 15 14 6
10 12 1 8
2 11 7
9 4
5

Also solved by Bob High, Harry Zaremba, Winslow Hartford, James Abbott, Mary Lindenberg, Allen Wiegner, Phyllis Grossberg, Alan Taylor, Larry Bell, Norman Spencer, Ken Rosato, Matthew Fountain, and the proposer.

N/D 3. Arthur Lewbel wants you to find the area of the small square without using Pythagoras's theorem. The area of the large square is one. The circle has center O, is tangent to two sides of the square, and passes through the center of the square.



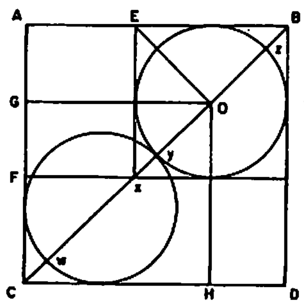
The following solution is from Tom Hansen (and son):

The area of the small square is 0.5, as the area of the small square is the same as the area outside the small square but within the large square.

To prove the answer, start with the figure shown in the question. Add a second circle with the same properties as the first circle, i.e. with the same radius (r), tangent to two sides of the large square, and passing through the center (Y). Draw a diagonal and add lines as shown on the enclosed figure, so that $\overline{BE} = \overline{AF} = 2r$ and $\overline{AG} = \overline{FG} = r$.

Look at the half of the large square defined by triangle ABC. Since the various angles involved are either 45° or 90°, it is easy to see that the area of region AEOG is equal to the area of region FGOX. Also, since $\overline{XY} = \overline{BZ} = \overline{CW}$, then $\overline{CX} = \overline{WY} = 2r$. Since $\overline{BE} = 2r$ as well, the areas of triangles BEO and CFX are equal. Therefore regions ABOG and CGO have equal areas. Similarly, regions BDHO and CHO also have the equal areas. The small square CGOH therefore has the same area as region ABDHOG, and since the area of the two regions together totals 1, the small square has area 0.5.

The same result can be obtained by trigonometry or by the proscribed theorem of Pythagoras. The correct result was also obtained by my ten-year-old son after 15 seconds of inspection, but he was just guessing. I think.



Also solved by James Cronander, Phelps Meaker, Winslow Hartford, James Abbott, John Morrison, Mary Lindenberg, Jules Sandock, Bob High, Harry Zaremba, William Kampe, Sidney Williams, Phyllis Grossberg, John McNear, Frederick Furland, Ken Rosato, Thomas Brendle, Larry Bell, Eugene Sard, Gordon Rice, Angel Silva, and the proposer.

N/D 4. Phelps Meaker wants to know the radius of a sphere circumscribing a regular tetrahedron.

Frederick Furland can help out Mr. Meaker:

As opposed to N/D 3, this problem can be solved inter alia through repetitions of Pythagoras's theorem.

The line through the center and perpendicular to the equilateral triangle forming the base of the regular tetrahedron is the locus of the center of all spheres whose surfaces touch the three vertices of that triangle. At one point on that line the surface of the sphere also touches the vertex of the tetrahedron. That is the sphere that we seek.

Figure 1. shows the base triangle, ABE of the tetrahedron. We find C, the center of the triangle as follows:

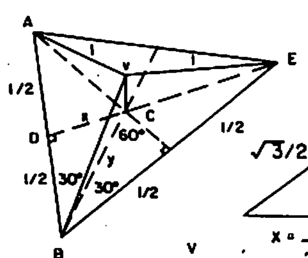


Figure 1

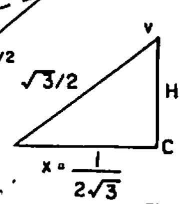


Figure 2

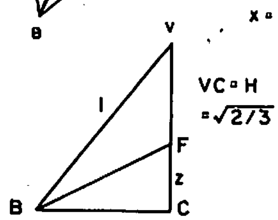


Figure 3

Each side of the triangle is of length 1. The three angle bisectors are perpendicular to the opposite sides and meet at the center of the triangle, C. Using the auxiliary 30° – 60° – 90° triangles shown in Figure 1, we find $x = 1/(2 \cdot 3^{0.5})$, $y = 1/3^{0.5}$.

Now, using one of the equilateral triangle sides of the tetrahedron, ABV, we can find the height of the tetrahedron, CV, from Figure 2. to be $H = (2/3^{0.5}$.

Using Figure 3, we see that the desired radius is BF = VF. Letting CF = z, we have

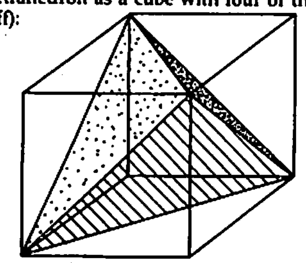
$BF = [z^2 + (1/3^{0.5})^2]^{0.5} = VF = (2/3)^{0.5} - z$

Solving $-z = 1/6\,(3/2)^{0.5}$. Thus, the radius of the sphere

$r = (2/3)^{0.5} - 1/6\,(3/2)^{0.5} = 0.612+$ times the length of a side of the tetrahedron.

A different technique was employed by Geoffrey Landis, who writes:

I don't usually respond to Puzzle, but problem N/D 4 has such an elegant geometrical solution that I felt compelled. The solution comes from constructing a cube by placing a plane onto each of the six edges of the tetrahedron (or equivalently, viewing the tetrahedron as a cube with four of the corners cut off):



Clearly, the circumscribing sphere is the same for both the cube and the tetrahedron, and has a diameter $\sqrt{3}$ times the edge length of the cube. Since the tetrahedron has an edge length $\sqrt{2}$ times the edge length of the cube, if the tetrahedron edge length is defined as "t" the radius of the sphere must be $(\sqrt{3/2})/\sqrt{2})t = \sqrt{6}/4\ t$.

Also solved by Jules Sandock, Robert Moeser, Avi Ornstein, Winslow Hartford, Bob High, Harry Zaremba, Steven Feldman, Ken Rosato (who thinks we may have asked this question before), David Wagger, Jim Landau, Daniel Morgan, Larry Bell, and the proposer.

N/D 5. Our final regular problem is from Gordon Rice who notes that, by successive flips of a coin, we may generate a random number between 0 and $2^N - 1$. Each flip determines one digit in the binary representation of the number; heads it's a one, tails it's a zero. Suppose we have a biased coin, which comes up heads with probability P. What is the expected value of an N-bit number generated with such a coin?

The key point is to notice, as did Steven Feldman, that the individual bits are independent so the expected value can be computed as the sum of the expected values due to each bit. Hence the answer is

$1P + 2P + 4P + \ldots + (2^{N-1})P = (2^N - 1)P$

Also solved by Geoffrey Landis, Bob High, Alan Taylor, Allen Wiegner, Daniel Morgan, Larry Bell, Winslow Hartford, and the proposer.

Better Late Than Never

1989 JUL 2. Larry Bell has responded.

JUL 3. Larry Bell has responded.

N/D SD1. Jim Landau points out that the author is really Greg DeStefano and Edwin Kruse points out that the year 2000 won't be as bad as predicted.

Proposers' Solutions to Speed Problems

SD 1. 2, 4, 5, 8, and 11.

SD 2. 81. Each of the 13 cards has a diamond in the upper left corner and one in the lower right corner, for a total of 26. The face cards have no other diamonds on them. The ace through 10 have 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = triangular 10 = 55 diamonds. 55 + 26 = 81.