ALLAN J. GOTTLIEB, '67

# The Quickest Way to Smother the Mate

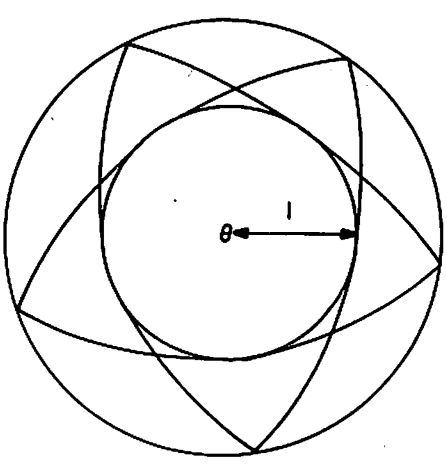
This being the first issue of another year, we again offer a "yearly problem" in which you are to express small integers in terms of the digits of the new year (1, 9, 8, and 9) and the arithmetic operators. The problem is formally stated in the "Problems" section, and the solution to the 1988 yearly problem is in the "Solutions" section.

## Problems

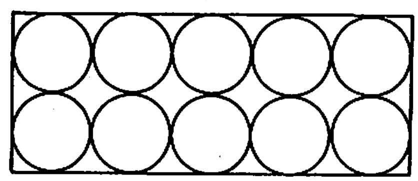
**Y1989.** Form as many as possible of the integers from 1 to 100 using the digits 1, 9, 8, and 9 exactly once each and the operators +, −, × (multiplication), / (division), and exponentiation. We desire solutions containing the minimum number of operators; and, among solutions having a given number of operators, those using the digits in the order 1, 9, 8, and 9 are preferred. Parentheses may be used for grouping; they do not count as operators. A leading minus sign *does* count as an operator.

**JAN 1.** Robert Bart wants to know the shortest chess game that ends in a true smothered mate, i.e., only the square the king is on is under attack, all the adjacent squares are blocked by "friendly" forces.

**JAN 2.** A non-Satanic pentagram (see diagram) is formed by intersecting five circular arcs evenly spaced around a circle. These arcs are tangent to a radius 1 circle concentric to the first and having half its area. Ken Rosato wants to know the radius of the arcs.




**JAN 3.** As illustrated below, it is easy to put $2n$ unit diameter circles inside a $2 \times n$ rectangle. Nob Yoshigahara and J. Akiyama want to know the smallest value of $n$ for which you can fit $2n + 1$ circles?




**JAN 4.** John Rule is interested in perfect squares that when written (in base 10) use all ten digits once each. What is the smallest such number? What is the largest?

## Speed Department

**SD 1.** Robert Dorich wants to know what the following numbers represent: 1345 and 11DE784A.

**SD 2.** Walter Cluett needs to extend the sequence 1427256.

## Solutions

**Y1988.** The problem was the same as Y1989, above, except that the digits to be used were 1, 9, 8, and 8.

The following solution is from John Drumheller. The double eights make the problem harder (and the double nines that we will soon encounter are also troublesome). Indeed, this difficulty will remain until 2013.

1 $1^{988}$
2 $1 + (9 - 8)^8$
3 $91 - 88$
4 $(9/18) \cdot 8$
5 -
6 $8 - (18/9)$
7 $8 - 1^{98}$
8 $89 - 81$
9 $1^{98} + 8$
10 $(18/9) + 8$
11 $88/(9 - 1)$
12 -

*SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.*

13 -
14 98 / (8 - 1)
15 (8 + 8) - $1^9$
16 (18 * 8) / 9
17 98 - 81
18 19 - (8 / 8)
19 19 + 8 - 8
20 19 + (8 / 8)
21 -
22 -
23 -
24 9 + 8 + 8 - 1
25 (1 * 9) + 8 + 8
26 1 + 9 + 8 + 8
27 91 - (8 * 8)
28 -
29 -
30 -
31 -
32 -
33 -
34 -
35 19 + 8 + 8
36 -
37 -
38 -
39 -
40 -
41 -
42 -
43 -
44 -
45 (8 * 8) - 19
46 -
47 (8 * (8 - 1)) - 9
48 -
49 -
50 -
51 -
52 -
53 -
54 (9 * 8) - 18
55 ((8 - 1) * 9) - 8
56 1 - 9 + (8 * 8)
57 -
58 -
59 -
60 -
61 -
62 -
63 (9 * 8) - 8 - 1
64 81 - 9 + 8
65 (1 + (9 * 8)) - 8
66 -
67 -
68 -
69 88 - 19
70 -
71 89 - 18
72 (81 / 9) * 8
73 (1 * 9) + (8 * 8)
74 1 + 9 + (8 * 8)
75 91 - 8 + 8
76 -
77 -
78 88 - 1 + 9
79 (88 * 1) - 9
80 98 - 18
81 (9 - 8) * 81
82 81 + 9 - 8
83 19 + (8 * 8)
84 -
85 -
86 -
87 88 - $1^9$
88 (19 - 8) * 8
89 $1^9$ + 88
90 (1 * 98) - 8
91 1 + 98 - 8
92 (8 / 8) + 91
93 -
94 -
95 -
96 88 - 1 + 9
97 (1 * 9) + 88
98 1 + 9 + 88
99 $1^8$ + 98
100 -

Also solved by Greg Spradlin, Harry Zaremba, Avi Ornstein, Steven Feldman, and Allen Tracht.

A/S 1. David Evans has placed white knights on a1, b1, and c1 and black knights on a4, b4, and c4. He wants you to find the minimum number of moves needed to interchange the positions of the knights disregarding possible captures. Only the first four ranks and the first three files are to be used.

The answer appears to depend on whether the movement of white and black knights must alternate. This was the proposer's intent (at least his solution has this property) and it seems right to me, since otherwise why have the knights colored at all? Requiring alternation, John Chandler found a solution with each color moving 9 times, or 18 total moves.

| | |
|---|---|
| 1. c1-a2 c4-a3 | 6. a1-c2 a4-c3 |
| 2. a2-c3 a3-c2 | 7. c2-b4 c3-b1 |
| 3. b1-a3 b4-a2 | 8. a2-c3 a3-c2 |
| 4. a3-c4 a2-c1 | 9. c3-a4 c2-a1 |
| 5. c3-a2 c2-a3 | |

Richard Hess submitted the following 16 (total) move solution and reports that Reiter proved this to be best possible (*Journal of Recreational Mathematics* 16(1), p. 7, 1983-84).

| | |
|---|---|
| 1. a4-c3 c3-a2 | 5. a3-c4 b4-c2 |
| 2. b1-c3 c3-a4 | 6. c2-a1 a2-c3 |
| 3. c4-a3 a3-b1 | 7. c1-a2 a2-b4 |
| 4. a1-c2 c2-a3 | 8. c3-a2 a2-c1 |

Also solved by Chris Unger, Jonathan Aronson, Matthew Fountain, Bill Habeck, and the proposer.

A/S 2. Matthew Fountain reports that a computer expert wanted to find the average length obtained for the largest part of a line of unit length when the line is randomly divided into four parts. The expert wrote a program that summed four random numbers between zero and one and divided the largest of these four random numbers by their sum. Is the average result he obtained from his program the length he sought?

This problem hinges on what is meant by randomly divided. Some readers agree with Chris Unger that the answer to the question posed is "yes or no." Unger points out that Martin Gardner asked a question concerning a randomly selected chord of a circle that admitted several solutions depending on the interpretation of randomly selected.

Other readers believe that the meaning of randomly divided is clear and that the computer expert did it wrong. The most convincing of these arguments, exemplified by Stephen Goldfeld's submission reprinted below, considers the simpler problem where the line is to be divided into only two parts. Tom Harriman, a member of this second camp, supplied a derivation (which may be obtained from the editor) that the correct average value for the largest piece is .52. Goldfeld writes:

It is easy to see that proposed procedure is *incorrect*, although it is rather messy to calculate analytically the size of the error. The source of the problem can be seen most simply if we restrict attention to the case of two random intervals. The proper answer is the expected value of the max of (r, 1 − r) where r is uniformly distributed on (0, 1). This problem has answer 3/4—loosely speaking, half the time the variable r would be ≥ 1/2 and have 3/4 as a mean, while the other half of the time (1 − r) will have 3/4 as a mean.

The proposed procedure generates, in our simplified setting, two uniform random variables, r1 and r2, and simulates the expected value of max [r1/(r1 + r2), r2/(r1 + r2)]. This yields a different answer, since r1/(r1 + r2) is no longer uniformly distributed on (0, 1). Indeed, it is more likely that this ratio will be in the neighborhood of 1/2 than it will be in the tails. As a consequence, the maximum will be less than the proper answer, 3/4, since the maximum calculated improperly will be closer to 1/2 more often than it should be. In the case of two intervals, the improper method yields an answer of about .693.

Also solved by Richard Hess, Matthew Fountain, Steve Feldman, Meredith Warshaw, Charles Whiting, Bill Habeck, and John Chandler.

A/S 3. Scott Berkenblit poses a challenge he saw in a Russian book of math problems. Find the exact value of the product

tan(80) tan (40) tan (20),

where all angles are expressed in degrees

The following solution is from Harry Zaremba: The exact value of the product of the tangents is $\sqrt{3}$. In proving this, it is noted that from the tangent of the sum of two angles,

$$\tan(40 + 20) = \frac{\tan(40) + \tan(20)}{1 - \tan(40)\tan(20)} = \sqrt{3}$$

which yields,

$$\tan(40) = \frac{\sqrt{3} - \tan(20)}{1 + \sqrt{3}\tan(20)}.$$

Also, from the trigonometric identity,

$$\tan(40) = \frac{2\tan(20)}{1 - \tan^2(20)}.$$

Equating the expressions for tan(40) and simplifying,

$$\tan^3(20) - 3\sqrt{3}\tan^2(20) - 3\tan(20) + \sqrt{3} = 0.$$

The equation above indicates that tan(20) is a real root of the cubic polynomial,

$$x^3 + Px^2 + Qx + R = 0$$

in which $P = -3\sqrt{3}$, $Q = -3$, and $R = +\sqrt{3}$.

The solution of the cubic results in three real roots which are,

$x_1 = 4\cos(10) + \sqrt{3} = \tan(80)$

$x_2 = 4\cos(130) + \sqrt{3} = \tan(-40)$

$x_3 = 4\cos(250) + \sqrt{3} = \tan(20)$

It is recalled that the product of the roots on an nth degree polynomial is equal to $(-1)^n A_n A_o$, in which $A_n$ is the constant term, and $A_o$ is the coefficient of the nth degree term. In the cubic, $n = 3$, $A_n = R = \sqrt{3}$, and $A_2o = 1$. Hence, the product of the roots of the cubic is,

$$\tan(80)\cdot\tan(-40)\cdot\tan(20) = (-1)^3\cdot\sqrt{3}$$

or,

$$\tan(80)\cdot\tan(40)\cdot\tan(20) = \sqrt{3}$$

Also solved by Richard Hess, Matthew Fountain, Thomas Harriman, Steve Feldman, N.F. Tsang, Phelps Meaker, Meredith Warshaw, Richard Williams, Jonathan Aronson, Bill Habeck, Daniel Morgan, Chris Unger, Stephen Goldfeld, Peter Silverberg, Ken Rosato, John Chandler, Frank Carbin, Charles Whiting, and the proposer.

A/S 4. Ken Rosato's rocket accelerates from 0 velocity to C (the velocity of light, 186,000 miles per second) with a constant acceleration (relative to a stationary observer) of 1g = 32 feet per second². It carries a clock synchronized to an identical clock at rest with the stationary observer. When the velocity of the rocket reaches that of light, how far behind the stationary clock will the clock on the rocket be.

Bill Habeck realized that if we let $t$ be the time in seconds then the difference in clock readings is

$$\frac{1}{g}\int_0^C 1 - \sqrt{1 - (gt/C)^2}\,dt.$$

He then evaluates the integral, using the substitutions $r = gt$ and $dr = g\,dt$, and obtains the answer $(4 - \pi)C/4g$.

Substituting for C and $g$ yields 6,586,130 seconds or 76 days, 5 hours, 28 minutes, and 50 seconds.

Also solved by Richard Hess, Matthew Fountain, John Prussing, Chris Unger, John Chandler, and the proposer.

A/S 5. Our last regular problem comes from the February 1986 issue of *IEEE Potentials*, where it was attributed to Bruce Layman. An IEEE student entered the north end of a tunnel of length L. After walking the distance L/4 into the tunnel, he noticed a car approaching the north entrance at 40 miles per hour. The student knew his own speed and calculated that no matter which end of the tunnel he ran to, he would arrive there at the same time as the car. What is his top speed? Hint: he might do better as a professional athlete than as an engineer.

Gordon Rice sent us the following solution:

Let S be the student's speed, and D the distance from the car to the tunnel entrance. The basic relation is time = distance/speed.

To reach the near end of the tunnel, the student takes L/4S and the car takes D/40. To reach the far end, the student takes 3L/4S and the car takes (L + D)/40. Solving L/4S = D/40 (thus D = 10L/S) and 3L/4S = (L + D)/40, the L cancels out and we get S = 20 mph.

Presumably the tunnel is too narrow for the car to pass the student and too dark for the driver to see him in time to stop. The student's best chance to save his life is to run for the far end. As they approach the exit, the running student will be silhouetted against "the light at the end of the tunnel"; the driver will perceive him, brake, and maybe stop and give him a lift back to town.

Also solved by Theodosios Korakianitis, Richard Hess, Bill Habeck, Matthew Fountain, Jonathan Aronson, Harry Zaremba, Frank Carbin, John Chandler, Thomas Harriman, Ken Rosato, Gerard Weatherby, Stephen Goldfeld, Avi Ornstein, N.F. Tsang, Evan Klein, Phelps Meaker, A. Ostapenko, Raymond Gaillard, Thomas Lewis, Charles Whiting, Gardner Perry, Frederick Furland, Bill Cain, Richard Riley, Chris Unger, John Prussing, and Steven Feldman.

### Better Late Than Never

A/S SD2. Some readers believed that Archimedes could have determined the walker's speed as the person approached. However, stories about Archimedes imply that when he worked on mathematics he was oblivious to approaching armies, let alone a single stroller.

### Proposers' Solutions to Speed Problems

SD 1. The speed of sound 741 mph (when written in octal) and the speed of light 299792458 meters/sec. (when written in hexidecimal).

SD 2. 14272563125. $1^1\,2^2\,3^3\,4^4\,5^5$.