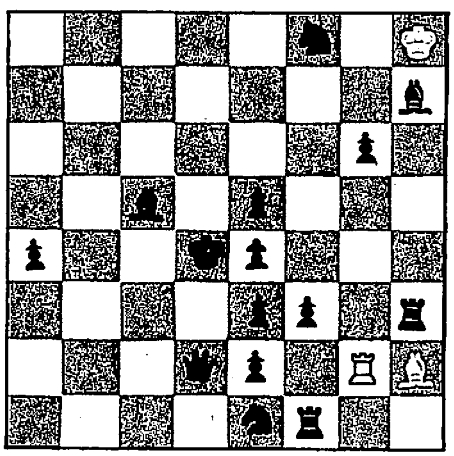
Mnage Solved. Now Try the Towers of Hanoi

It has been at least a year since I specified the size of the backlogs for all the various kinds of problems that are printed. Let me do so now. I have close to a year's supply of regular, chess, and speed problems; computer and bridge problems are in short supply.

I would like to thank all those readers who sent season's greetings (this is the first column I am writing in 1988) to my family and me. Alice, David, Michael, and I wish you all a healthy and happy new year.

APR 1. We begin with a helpmate problem from *The Tech*, M.I.T.'s student newspaper. Black moves first, and White and Black are to cooperate so that Black is mated on White's fourth move.

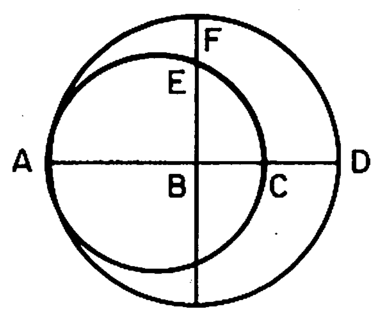


APR 2. John Rule wants you to find two positive integers differing by five such that the sum of their squares is a perfect cube, and to show that the solution is unique.

APR 3. Peter Gottlieb (no relation) has an interesting variation on the famous Towers of Hanoi problem:
As with the standard version, there are three pegs; initially there are n rings on one of the pegs, graded in size from largest at the bottom to smallest on top. The problem is to find the minimum number of moves to transfer all of the rings to another peg, one at a time, never having a larger ring on top of a smaller one. The variation is that only clockwise moves are permitted. For example, if the pegs are labeled A, B, C, then the only moves permitted are A to B, B to C, and C to A. There are two cases of solution, one for moving all the rings to the nearest peg in the clockwise direction, and one for moving to the third peg.

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

APR 4. Here is one Jim Landau found in the *Washington Post* classified ads:
The two circles shown in the diagram below touch at A. The larger circle has its center at B. The width of the crescent between points C and D is 90 mm. and between points E and F the distance is 50 mm. What are the diameters of the two circles?



APR 5. Ronald Martin has a magic square problem for us:
Arrange the integers from 1 to 25 in a 5 × 5 grid so that each column and row sums to 65. Moreover, each of the 10 diagonals is to sum to 65. These 10 diagonals are formed by starting at each of the 5 elements in the first row and proceeding southwest and southeast, identifying the left and right edges of the grid. For example, one diagonal includes the elements located at (1,4), (2,3), (3,2), (4,1), (5,5).

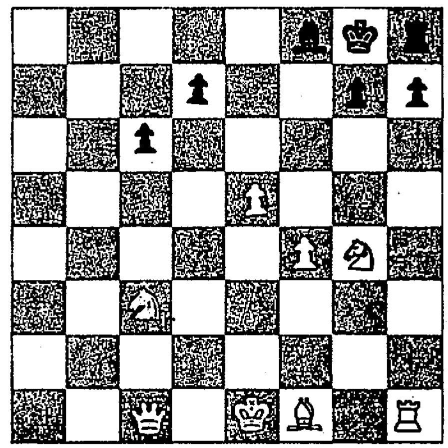
Speed Department

SD 1. Jim Landau wants to know: What is the rarest play in chess?

SD 2. Charles Piper has a glass of wine and a glass of water. He takes a teaspoon of the wine and stirs it into the water and then takes a teaspoon of the result and puts it back into the wine. Is there more water in the wine or more wine in the water?

Solutions

DEC 1. White is to move and mate in two.



C. Larson found the following solution:
1. B–c4 ch P–d5
2. P×P e.p. mate

Matthew Fountain notes that *Chess Life* reported that a similar theme has been used in a TV spy movie and in its Russian remake.

Also solved by David Simen, Steven Feldman, Karim Roshd, Brian Lebowitz, Steven Gordon, Dennis White, Scott Berkenblit, Richard Hess, Kenneth Bernstein, Ron Raines, John Cronin, Avi Ornstein, Elliott Roberts, Abraham Fineman, Turner Gilman, Mark Haberman, Michael Jung, Paul Herkart, Allan Orloff, Tomo Hasegawa, and the proposer.

DEC 2. Determine, without using a computer or calculator, which is larger, e^π or π^e.

The following solution is from R.P. Boas:
Here is the shortest proof I know that $\pi^e < e^\pi$:
For $x > e$, $x/\ln x$ increases (its derivative is positive), so
$x/\ln x < y/\ln y$ if $y > x \geq e$.
$x \ln y < y \ln x$
$e \ln \pi < \pi \ln e$
$\pi^e < e^\pi$.

Also solved by Kenneth Bernstein, Jim Landau, David Wagger, Winslow Hartford, Richard Hess, Scott Berkenblit, Dennis White, Steven Gordon, Brian Lebowitz, Karim Roshd, David Simen, Robert Cesari, Alan Taylor, Harry Zaremba, Bill Cain, Edward Dawson, John Moore, Christopher Grayce, Ken Haruta, Mary Lindenberg, John Prussing, Benjamin Wurzburger, Ken Rosato, Timothy Maloney, Frank Quinn, and Matthew Fountain.

DEC 3. Show that there are no positive integer solutions to:
$X^3 + Y^3 + 4 = Z^3$.

The following solution is from Edward Dawson:
The cube of any integer is given by one of the following functions of an arbitrary integer N:
$(9N + 4)^3 = 9(81N^3 + 108N^2 + 48N + 7) + 1$
$(9N + 3)^3 = 9(81N^3 + 81N^2 + 27N + 3)$
$(9N + 2)^3 = 9(81N^3 + 54N^2 + 12N + 1) - 1$
$(9N + 1)^3 = 9(81N^3 + 27N^2 + 3N) + 1$
$(9N)^3 = 9(81N^3)$
$(9N - 1)^3 = 9(81N^3 - 27N^2 + 3N) - 1$
$(9N - 2)^3 = 9(81N^3 - 54N^2 + 12N - 1) + 1$
$(9N - 3)^3 = 9(81N^3 - 81N^2 + 27N - 3)$
$(9N - 4)^3 = 9(81N^3 - 108N^2 + 48N - 7) - 1$

These equations show that no cube of an integer differs by more than 1 from an integral multiple of 9. So when the sum of any two cubes of integers is subtracted from the cube of a third integer, the result cannot differ by more than 3 from an integral multiple of 9. Therefore no solution in positive integers is possible for either of the equations $X^3 + Y^3 + 4 = Z^3$ and $X^3 + Y^3 + 5 = Z^3$.

Also solved by Steven Gordon, Scott Berkenblit, Richard Hess, Winslow Hartford, David Wagger, Jim Landau, and Kenneth Bernstein.

DEC 4. What is the probability that six random numbers chosen (without replacement) from 1 to 36 will have at least one adjacent pair (like 12 and 13, for example).

The following solution is from Matthew Fountain: The probability for at least one adjacent pair is 0.621992. We begin with our solution by constructing an array in which $a_{i,j}$ is the number of ways j integers can be assigned positive integer values equal to or less than i, with no two values adjacent or equal. $a_{i,j} = 0$ when $i < 2j-1$, as j integers not adjacent to each other require a range of at least $2j-1$. $a_{i,1} = i$, as in this case the one integer may assume any value from 1 to i. When $j > 1$, $a_{i,j} = a_{i-1,j} + a_{i-2,j-1}$. This is true because $a_{i-1,j}$ is the number of ways the j integers fit in the range 1 to $i-1$ without being adjacent and $a_{i-2,j-1}$ is the number of ways j integers fit in the range 1 to i when one integer equals i. Fortunately, only a small part of the array needs to be constructed. The following is more than enough.

1	2	3	4	5	6	7	8	9	10	11	12
	0	1	3	6	10	15					
			0	1	4	10	20				
					0	1	5	15			
							0	1	6	21	
									0	1	7

The bottom row of the array indicates that six non-adjacent digits fit in the range 1 to 10 in 0 ways, in the range of 1 to 11 in 1 way, and in the range of 1 to 12 in 7. These simple cases are easily verified. The differences $1-0=1$ and $7-1=6$ are in the row above. In fact, due to our method of construction, each row is the differences of the row below, and the differences in the top row are all equal to one. Therefore we can extrapolate with Newton's interpolation formula to find the value $a_{36,6}$. Taking $a_{10,6}=0$ as the base for our extrapolation, the successive-order differences are 1,5,10,10,5,1 and $a_{36,6} = 0+26(1+(25/2)(5+(24/3)(10+(23/4)(10+(22/5)(5+21/6))))) = 736281$.

The number of ways taking 6 values from 36 is $36!/(6!30!) = 1947792$. The probability of at least one adjacent pair is $1 - 736281/1947792 = 0.621992$. The differences 1, 5, 10, 10, 5, 1 are the binomial coefficents of $(1+a)^5$. Inspection of the array reveals that the values in third, fourth, and fifth rows can similarly be extrapolated using differences that are binomial coefficients. A general formula for any $a_{i,j}$ could be derived from this observation.

Several readers used computer searches to arrive at their solutions. Robert Moeser makes the observation that he first wrote a straightforward brute-force program and then derived a clever version that avoided much of the searching. However, this latter program ran slower! His conclusion: "So much for cleverness."

Also solved by Kenneth Bernstein, Steven Gordon, Winslow Hartford, Mary Lindenberg, David Simen, Harry Zaremba, Dennis White, Steven Feldman, Richard Hess, Simson Garfinkel, and Scott Berkenblit.

DEC 5. Solve the Mnage problem, where you have a certain number of couples to dinner and you wish to seat them at a round table, with men and women alternating and all the couples separated. How many different arrangements are possible for three couples? For four? For ten?

The following solution is from Harry Zaremba: To generalize, assume there are n couples invited to a dinner, and that the position of one woman at the table is considered fixed as a reference point. In alternate locations from the fixed point there will be $(n-1)$ chairs available to the same number of women. The first alternate chair clockwise from the fixed position can be selected by any one of $(n-1)$ different women, the second alternate chair by one of $(n-2)$ different women, the third chair by one of $(n-3)$, and so on until the last or $(n-1)$st chair remains for one woman. In all there will be $(n-1)!$ different permutations in which the women can be seated at the table. Now with each of these permutations, a chair between any two women can be occupied by any one of $(n-2)$ different men, none of whom is an escort of either woman. Hence, the total number of different seating arrangements possible to fulfill the problem requirements for n couples will be,

$P_n = (n-1)!(n-2)$.

The number of arrangements when n = 3,4, and 10 are $P_3 = 2! \times 1 = 2$, $P_4 = 3! \times 2 = 12$, and $P_{10} = 9! \times 8 = 2{,}903{,}040$, respectively.

Also solved by Mary Lindenberg, Matthew Fountain, Winslow Hartford, Richard Hess, and the proposer.

Better Late Than Never

JUL 2. Edward Dawson notes two misprints in the published version of his solution. The vertex at the lower left should have three coordinates, the second of which is $-\sqrt{3}u/2$ and in the text Y-coordinates have numerators of $\sqrt{3u}$ instead of $\sqrt{3}u$.

JUL 3. Michael Jung and Turner Gilman note that the span between 15:55:51 and 20:00:02 is 4:04:11.

JUL 5. Michael Jung points out that there are actually three triangles in the drawing. Moreover, the shaded region has area $\theta - .5\sin\theta\cos\theta$.

OCT 3. Avi Ornstein and John Cronin have responded.

OCT 4. Raymond Gaillard has responded.

OCT 5. Benjamin Wurzburger has responded.

Proposers' Solutions to Speed Problems

SD 1. Underpromotion to a bishop or rook. The only plausible reason for such underpromotion is to prevent a stalemate, as illustrated in problem 1984 F/M 1.

SD 2. Neither. They are the same.