ALLAN J. GOTTLIEB

How Smith Tried (and Failed) to Vex Brown

Well, as this column is written it's two for two for the big apple. Congratulations to the NY/NJ Giants on their victory in the superbowl. Of course, by the time you read this the hockey and basketball seasons will be over and we will be two for four.

Problems

APR 1. We begin with a double-dummy bridge problem from Doug Van Patter who writes:

South is declarer at a six-heart contract. Opening lead: ♠5. Make 12 tricks against best defense.

North:
♠ A K 10 4
♥ A J 10
♦ A 7 5
♣ A 5 3

West:
♠ Q 9 8 5
♥ 2
♦ J 9 4 3 2
♣ Q 9 2

East:
♠ J 3
♥ Q 9 4 3
♦ 10 8 6
♣ J 8 6 5

South:
♠ 7 6 2
♥ K 8 7 6 5
♦ K Q
♣ K 10 4

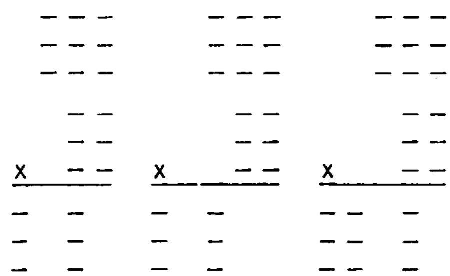
This hand occurred at a duplicate bridge club. My partner was the only declarer to make 12 tricks. In her successful line of play, East made the ♥9 at the end of the hand. After looking at this double-dummy problem, another friend suggested an entirely different approach. In his line, East makes the ♥9 earlier in the play. What are these two successful lines of play?

APR 2. J. Ruoff has three circles of different radii and no circle contained within another. He would like you to construct a fourth circle tangent to the other three.

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

APR 3. Nob. Yoshigahara reports that the liquid crystal display on his calculator cannot draw vertical lines. This leads to an interesting variant of cryptarithmetic problems:

The diagram represents the product of three-digit numbers by two-digit numbers as displayed on Mr. Yoshigahara's calculator. What were the original problems?



APR 4. A stellar problem from Phelps Meaker:

A star-shaped figure can be drawn with two concentric circles, drawing chords tangent to the inner circle, successively around the outer one until they meet the starting point. How many points, blunt as they may be, result if the inner circle is .66913 of the outer? Compass and straightedge not allowed.

APR 5. Our final regular problem is from the late John Rule:

Mr. Brown asked Mr. Smith to perform the following operations in the order named, without Brown's being able to see Smith's work:

(a) Write any positive integer, preferably of two digits to save labor on the part of Smith.
(b) Multiply this number by the next highest integer.
(c) Multiply the result of (b) by 225.
(d) Add 56 to the result of (c).
(e) Tell Brown all the result of (d) except the two right-hand digits.

Smith gave 4064 in response to the request in (e), whereupon Brown, after a moment's computation, informed Smith that his result after step (d) was 406406 and that the number he originally chose was 42. Smith confirmed these statements. How did Brown reach his conclusion?

Speed Department

SD 1. Jim Landau wants to know what is small, yellow, and equivalent to the Axiom of Choice.

SD 2. Howard Sard asks us to construct a bridge deal such that North-South can win 13 tricks in a suit contract against best defense holding the minimum number of high-card points.

Solutions

N/D 1. In a duplicate bridge tournament in which every hand was played 20 times, the result at four tables was a final contract of one club, played once from each of the four sides. At four of the other tables it was played at one diamond once from each side. Likewise, at the remaining tables it was played once from each side at one heart, one spade, and one no-trump. Every one of these contracts was set. Analysis of the hand proved that none of the declarers made a mistake in play. Unfortunately, Mr. Kells failed to see what the deal was. Can you reconstruct it? (That is, a deal where any contract anybody bids can be set no matter how hard the successful bidder tries to make it.)

The following solution is from the proposer, Lawrence Kells:

	♠ 8 7 6 ♥ A K Q 5 4 3 2 ♦ J 10 9 ♣ —	
♠ — ♥ 8 7 6 ♦ A K Q 5 4 3 2 ♣ J 10 9		♠ A K Q 5 4 3 2 ♥ J 10 9 ♦ — ♣ 8 7 6
	♠ J 10 9 ♥ — ♦ 8 7 6 ♣ A K Q 5 4 3 2	

Because of the symmetry of the hands, we need only consider what happens to South at the five possible contracts. At one club, West takes ♦A and ♦dcK, then gives East a diamond ruff. East takes ♠A, ♠K, and ♠Q and leads a fourth spade, promoting a trump in West's hand as the seventh defensive trick. At one heart, West takes ♦A, ♦K, and ♦Q and keeps leading diamonds until dummy ruffs, promoting a trump trick for East. East will also win any spades still in the dummy, resulting in six side-suit tricks for the defense, as well as the one trump trick. At one spade, West takes ♦A and ♦K, then gives East a ruff. East cashes six more spades for a three-trick set. At one diamond or one no-trump, West simply takes seven diamond tricks.

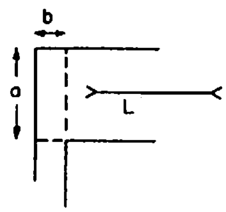
Also solved by Winslow Hartford, Matthew Fountain, and David Smith. Thomas Harriman has responded.

N/D 2. A certain polyhedron has nine vertices, and each of its faces is a triangle. How many faces does the figure have? If six faces meet at each of three vertices, what common number of faces meet at each of the other vertices?

Albert Mullin sent us the following solution:
We use Euler's Theorem on polyhedral surfaces in 3-space: $e + 2 = f + v$. Note that this formula holds for polyhedra that may *not* be either regular or even convex! Indeed, an analogous formula holds for polyhedra that cannot be continuously deformed into a sphere. For example, on a *torus* a polyhedron satisfies the relation: $f + v - e = 1$. Analogous formulae hold for polyhedra with genus > 1, too. Further, Poincare generalized such formulae from 3-space to n-space. Counting edges, we have three (per face), but this process counts each edge twice; hence, by Euler's formula:
$3f/2 + 2 = f + 9$;
$f/2 = 7$
$f = 14$ (faces).
So the polyhedron has: $e = 21$ (edges), $f = 14$ (faces) and $v = 9$ (vertices). Now count edges, based on the information that six faces meet at each of three vertices, noting that each edge is common to *two* vertices. Thus, $2 \times 21 = 6 \times 3 + x \times 6$ or $x = 4$ (faces at each of their vertices). That is, *four* faces meet at each of the other six vertices.

Also solved by Oren Cheyette, Winslow Hartford, James Landau, Charles Piper, Chip Whiting, Avi Ornstein, Matthew Fountain, Thomas Harriman, and the proposer, Harry Zaremba.

N/D 3. What is the maximum length board L that can pass through the corner shown?



David Smith can move pianos with the best of them:
The length of the line passing through the intersection of the inner walls, terminating at its intersection with the outer walls and making an angle ϕ with the horizontal wall in the diagram, is $L = a/\sin\phi + b/\cos\phi$. The minimum value of L, which is the maximum length of board that can pass through, is obtained by setting $dL(\phi)/d\phi = 0$, from which $a/b = \tan^3\phi_m$. Thus $\phi_m = \tan^{-1}(a/b)^{1/3}$ and the maximum length board that can get around is $L_m = a/\sin\phi_m + b/\cos\phi_m$.

Joel Kalman notes a rule of thumb that the maximum length board that can be maneuvered around a corner joining two hallways of equal width is $L = 2.8W$.

Also solved by Erik Borne, Winslow Hartford, Martin Carrera, Stu Lerner, Oren Cheyette, Michael Jung, Jim Landau, Howard Lyons, Mary Lindenberg, Steve Feldman, Chip Whiting, Charles Piper, Robert Moeser, Shawn Gaither, Stuart Kurtz, Harry Zaremba, Matthew Fountain, Thomas Harriman, and the proposer, Rubin Cohen.

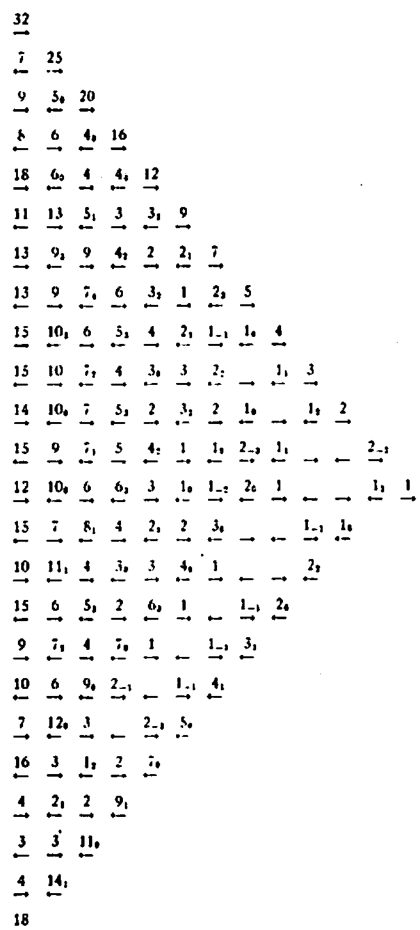
N/D 4. Any one of a group of aircraft may be refueled from any other aircraft. Each has a fuel capacity sufficient for a flight one-fifth the distance around the earth. Assuming that all have the same constant ground speed and the same rate of fuel consumption, that the only landing place and the only available fuel supply are at the home base, and that refueling time is negligible, find the minimum number of planes necessary so that one plane may fly around the earth and all return home safely.

The best solution that I have seen is the one in the *American Mathematical Monthly* that was mentioned when the problem was posed by Albert Mullin. This solution, from the original proposer Fred Jamison, requires just 75 planes. We reprint the solution without change from the April 1951 issue:
The minimum number of planes required is not more than 75.

It seems reasonable that a sufficient number of planes would depart from the home base so that, by refueling, one plane would be fueled to capacity when two-fifths of the distance around the earth and be met there by a plane flying in the opposite direction. Let two-fifths of the distance be divided into 12 equal legs. (Each leg is 1/30th of the earth's circumference, and a plane fueled to capacity has enough fuel for six legs.) A schedule is given to show how 77 planes can accomplish the desired feat. Thirty-two planes depart at the same time; at the end of the first leg 25 are fueled to capacity and 7 return; at the end of the second leg 5 turn back after refueling 20 to capacity; but at the same instant 9 depart from the home base, and so on. Let the amount of fuel necessary for one plane for one leg be called a portion. Subscripts on numbers of returning planes indicate portions of fuel remaining at the end of the leg, negative subscripts on numbers of outbound planes indicate portions of fuel below capacity at the beginning of the leg.

After six legs have been completed by the first flight of planes it would be necessary to send planes from the home base in the opposite direction to meet the plane flying around the earth. For this operation we have only to use the lines of the schedule in reverse order. The numbers of planes in the air at various times are now easily computed, and it is found that no more than 77 are required at any one time. However this number can be reduced by one if on the eighth outbound flight of 10 planes, one plane turns back at the midpoint of the first leg after refueling the other 9 planes (with a similar change in the analogous flight which is to meet the home-coming plane). A further reduction of one is possible if on the sixth, seventh, ninth and tenth flight each, one plane is turned back at the midpoint of the first leg; and if on the eighth flight one plane is turned back at the quarter point and one at the three-quarter point on the first leg.

It is reasonable to suppose that the minimum number demanded in the proposal is less than 75, since the schedule here presented makes the simplifying assumptions that changes of direction and refueling occur only when an integral number of legs have been traversed and that all planes flying at any instant are flying in the same direction (indicated by the arrow). Units of time are indicated vertically, distances (in legs) horizontally.



Also solved by Michael Jung, Thomas Harriman, and Matthew Fountain.

N/D 5. Find a method of converting an arbitrary (legal) position in a tower of Hanoi puzzle into another arbitrary position. In the tower of Hanoi puzzle, we have M disks of differing radii distributed on three pegs with no disk on top of a smaller one.

The problem of finding a *minimal* solution appears to be formidable. For example, it is clearly not always optimal to first move all the disks to one spindle. The proposer, Edmund Staples, first considers a "natural conjecture," which at first appears to be minimal; however, Staples gives an example that shows this to be nonminimal. The optimal solution is then derived. In detail, Mr. Staples writes:

The traditional tower of Hanoi problem asks how to transfer N disks (numbered 1,2, . . . N, smallest to largest) from spindle a to spindle b using a third spindle c as spare, the moves subject to well-known rules. In this note I discuss the computation of the optimal solution when the start and end positions are arbitrary distributions of disks over spindles. The solution given here uses a function L which is computed bottom-up rather than the completely top-down recursion of the usual problem.

If m(k,x,y) denotes the move of disk k, 1 ≤ k ≤ N, from spindle x to another spindle y with x,y = a,b,c, and if S(N,x,y) denotes the sequence of moves taking a tower of N disks from x to y (<> x), then the traditional solution is given by the recursive formula:

S(N,x,y) = S(N − 1,x,z) m(N,x,y) S(N − 1,z,y)

where the right hand side is a concatienation of sequences; x,y,z are a,b,c in some order; and S(O,x,y) consists of zero moves. If L(N,x,y) is the number of moves in S(N,x,y), then L(N,x,y) = $2^{(N-1)}$.

In seeking a solution to the extension, a very natural first conjecture is that the above solution generalizes with the proviso that the kth recursive step is skipped if disk k starts and ends in the same position. Precisely, let P,Q represent positions of N disks (N may vary with P or Q), let P(k) represent the position of the first k disks of P, let sP represent the spindle that disk N is on in P (disk N is the largest disk of P), let T(x,k) denote the tower, or position, where the first k disks are all on x, and let S(P,Q) denote a sequence of moves from P to Q. The natural conjecture is that an optimum solution is given recursively by:

S(P,Q) = S[P(N − 1),Q(N − 1)] if sP = sQ, or otherwise

S(P,Q) = S[P(N − 1),t] m(N,sP,sQ) S[t,Q(N − 1)]

where t = T(z,N − 1) and z <> sP or sQ. If N = 0 then S(P,Q) trivally contains no moves. This clearly is a solution; however it is *not* the optimum solution. Consider the positions with N = 3 and P has disks 1 and 2 on a, 3 on b; while Q has 1 and 2 on b, 3 on a. The solution just given yields the seven-move sequence:

S(P,Q) = m(1,a,b)m(2,a,c)m(1,b,c)m(3,b,a)m(1,c,a) m(2,c,b)m(1,a,b).

However, the optimum solution is the five-move sequence:

S = m(3,b,c)m(1,a,c)m(2,a,b)m(1,c,b)m(3,c,a).

Thus, the natural conjecture is wrong. We may be able to improve upon it. We shall see, however, that it comes very close to being correct. Note in the last solution that the bottom disk moved twice. Our essential observation will be that in any optimum solution, the bottom disk need never move more than twice. Adopting the notation:

S = S0 m1 S2 m2 S3 m3 . . . mj Sj

where Si = a sequence of moves of disks 1 . . . N − 1 and mi = a move of disk N. If sP = sQ, it is clear that mi's have no effect; and if j > 0 then

S' = S0 S1 S2 . . . Sj

is a solution with fewer moves. Thus, if S is an optimal solution from P to Q and sP = sQ, disk N is never moved. If sP <> sQ, then obviously j > 0. We must have j < 3, however, for otherwise disk N would move three or more times and repeat a spindle x. By the argument just given, we could drop the moves of disk N which are between the two visits to x. Next, note that if P is a position and its bottom disk N can be moved (say the move is m[N,x,y]) then clearly we must have P(N − 1) = T(N − 1,z) where z <> x,y. From this we may give a three-case representation to the optimal solution:

If sP = sQ (case A):

S(P,Q) = S[P(n − 1),Q(N − 1)], otherwise

S(P,Q) = S[P(N − 1),t]m(N,sP,sQ) S[t,Q(N − 1)] (case B), or

S(P,Q) = S[P(N − 1),t']m' S(t't")m"S[t",Q(N − 1)] (case C)

where m' and m" are appropriate moves t,t't" are towers. When sP <> sQ (case A), we must decide between case B and case C. This is easy to do if we can compute the function:

L(P,k,x) = number of moves in S[P(k),T(k,x)].

It is easy to see that L(P,k,x) ≤ $2^{(k-1)}$, for we can do no worse than the natural conjecture. If P is a tower distinct from T, then equality holds. Applying the representation to S[P(k),T(k,x)] and using this estimate allows us to rule out case C to obtain:

L(P,k,x) = L(P,k − 1,x) if sP(k) = x, or

L(P,k,x) = L(P,k − 1,y) + 2^k, where y <> x, sP(k).

Once the function has been computed the decision between case B and case C can be made. It is interesting to note that this decision needs to be made only once in any given problem, for the subsequent recursive steps will give a P or a Q (or both) which is a tower, so case C can then be immediately rule out as just noted. In teaching undergraduate data structures, I have found it an interesting programming exercise to implement this solution. These positions are best represented not as lists of disks for each spindle, as one might expect, but rather as an array whose kth entry is a pointer to the spindle that holds disk k.

Also solved by Oren Cheyette, Matthew Fountain, Robert Moeser, Thomas Harriman, and Winslow Hartford.

Better Late Than Never

Y1986. Rik Anderson has responded.

1986 JAN 2. Norman Wickstrand has improved his solution.

JAN 4. Thomas Brendle notes that the problem required calculating the ratio of the *three largest* circles' area to the sector's area. The largest ratio occurs at 21° and is approximately .7591.

F/M 4. Matthew Fountain and Harry Zaremba believe that the original published solution is correct. Mr. Zaremba writes:

The method of solution used by Mr. Goldstein is incorrectly applied and the velocity obtained is not in the direction requested by the problem. Mr. Hendrickson unfortunately has a serious flaw in his derivation of V and V_x: the expressions given do not have the dimensional units of velocity. Their units would result in square centimeters per second. When Y_0 = L/2 and Y = L/3, the correct expressions for V and V_x would be,

$V = L/Y \times V_x/2 = 3V_x/2$, and

$V_x = Y\sqrt{3g(Y_0 - Y)}/L = \sqrt{2gL/6}$.

When L = 100 cm and g = 980 cm/sec², the horizontal velocity of the rod's center of gravity is

$V_x/2 = \sqrt{2 \times 980 \times 100/12} = 36.9$ cm/sec

as given in the published solution.

A/S 1. Mark Seidel has found a simpler solution.

A/S 5. Alan Hodgkinson has responded.

OCT 2. Joel Feil has responded.

OCT 3. Joel Feil, John Cushnie, Mark Foster, and Mary Lindenberg have responded.

OCT 5. Joel Feil and Mary Lindenberg have responded.

Proposers's Solutions to Speed Problems

SD 1. Zorn's Lemon.

SD 2. Five high-card points suffice, as the following deal illustrates:

	♠ 10 9 8 7 6	
	♥ —	
	♦ 7 6 5 4	
♠ S K	♣ 7 6 5 4	♠ Q
♥ K Q J		♥ A 10 9
♦ A K Q J	♠ A J 5 4 3 2	♦ 10 9 8 3 2
♣ A K Q J 10	♥ 8 7 6 5 4 3 2	♣ 9 8 3 2
	♦ —	
	♣ —	

hand each time by trumping a diamond or club.

South makes seven spades against any defense by playing one round of trumps, then trumping three rounds of hearts in dummy, returning to his hand each time by trumping a diamond or club.