ALLAN J. GOTTLIEB

How to Solve Tokyo's Parking Problem

Matthew Fountain reports that the historical journal he and his wife edit has been well received. Over 100 new members have joined their historical society to obtain the journal, and a newspaper reporter is writing a story on it.

Our supply of computer-related problems is running low, and speed problems are in critically short supply. So keep those cards and letters coming!

Finally, I would like to dedicate this column to the seven shuttle astronauts killed while trying to advance our understanding of science and nature and to their families (including their large NASA family).

Problems

APR 1. We begin with a computer-related problem from Harry Zaremba:
Frequently it is useful to know what the date and date-of-week will be after a certain number of days will have elapsed. When the number of days is large, use of a calendar to establish the date is a bit awkward, and a calendar may be of little help when it is found that the required date falls into the range of a different year. For example, a convicted felon who is sentenced to the "pen" for 300 days after December 5, 1985, may be interested to know that his last day of imprisonment will be Wednesday, October 1, 1986. The problem therefore is to develop a program that determines the date and day-of-the-week for any Nth day after a selected date.

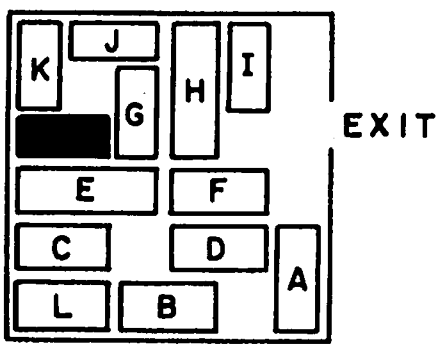
APR 2. Frank Rubin wants to introduce some friends of his, the family of functions

$$f_n(x) = \int_{t=0}^{x} \sin(t^n)dt$$

The infant f_1 is reasonably well behaved; it is bounded and oscillates between -1 and $+1$. How well behaved are the other family members, f_2, f_3, etc.?

APR 3. Emmet Duffy wants you to arrange the digits 1 through 9 into two proper fractions and two positive integers so that the four numbers total 100. One such arrangement is 1/8, 63/72, 94, and 5. Find seven others. The result of interchanging digits in the integers is not considered a different arrangement—e.g., 1/8, 63/72, 95, and 4 is considered the same as our original arrangement.

APR 4. An urban problem from Nob Yoshigahara.
The following is a parking area found in Japan. Your car is black. You want to go out. Because so many cars are packed in, many have to be moved before yours is free. In fact, so many cars are packed in that they can move only back and forth and cannot turn. How many moves do you need to get out from here? The fewer moves, the better.



APR 5. P.V. Heftler wants you to find the smallest number that can be partitioned into four distinct positive integers such that the sum of every pair is a perfect square.

Speed Department

SD 1. Phelps Meaker, whose eyes are 62 inches above the ground, notes that the floor of the aisles in his favorite supermarket is glossy linoleum and reflects the fluorescent lamps 10 feet above, spaced 6 feet apart and installed crosswise of the aisle. As he moves down the aisle at a uniform rate of 42 inches per second, the reflected images move as well. How often does he catch up with an image?

SD 2. Jim Landau asks for the significance of the sequence of numbers 2, 5, 5, 4, 5, 6, 3, 7, 6.

Solutions

N/D 1. Given the coordinates of four points, determine if they form the four corners of a square. No assumption can be made about the order in which the points are presented.

Jim Landau found a typographical error in the problem; in order to form a square, the third point must be (7,4), not (2,4). But he conquered this hazard to send us a sweet solution designed for both human and machine consumption. He describes the problem as "a piece of cake, or more exactly a batch of cookies. In a previous office, where we did much map reading, we called this problem the 'cookie cutter' because once we had specified the four points, we had cut out a quadrilateral from the rest of the map, just like cutting out a cookie from a sheet of cookie dough. It was interesting that, while it was not difficult to read four sets of coordinates from a map, it proved surprisingly difficult to read the four points in clockwise sequence, northwesternmost point first. Therefore we had to develop an algorithm to take four points in any order and try to make a convex quadrilateral out of them." Landau says there are a number of ways to solve the problem, based on various geometrical theorems. He thinks the following is the simplest:

A quadrilateral is a convex quadrilateral and in fact a rhombus if all four vertices are distinct and all four sides are equal. A rhombus is a square if both diagonals are equal.

Proof: Consider 3 points A, B, and C with AB = BC ≠ 0. Construct a pair of circles with centers A and C and with radii = AB. The circles will intersect in two points, B and a new point which we will call D. (If A, B, and C are collinear then points B and D will be the same.) Also A, B, C, and D are coplanar and CD = AD = AB = BC.

If B and D are distinct points then they are on opposite sides of the line AC. If A and C are distinct points then they are on opposite sides of the line BD. Hence the quadrilateral ABCD is convex if A, B, C and D are all distinct.

Triangles CBD and ADB are congruent because all three pairs of corresponding sides are equal. Hence angle CBD = angle ADB which means BC and AD are parallel. Similarly AB is parallel to DC. Hence ABCD is a parallelogram and, since all four sides are equal, a rhombus.

If the diagonals BD and AC are equal, then triangles ABD, CBD, BAC, and DAC are all congruent

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(because all three pairs of corresponding sides are equal); hence angle ABD = angle DAC, angle CBD = angle BAC, and by summing, angle BAD = angle ABC. By similar arguments, angle BAD = angle ADC = angle DCB, so all four vertex angles of the rhombus are equal. Since the angles of a quadrilateral add up to 360 degrees, then each vertex is 90 degrees and ABCD, with four equal sides and four right angles, is by definition a square, QED.

The algorithm is as follows: Given four coplanar points W, X, Y, and Z (not necessarily distinct),
if WX = XY = YZ = ZW ≠ 0 and WY = XZ ≠ 0, then WXYZ is a square; or
if WX = XZ = ZY = YW ≠ 0 and WZ = XY ≠ 0, then WXZY is a square; or
if WY = YX = XZ = ZW ≠ 0 and WX = YZ ≠ 0, then WYXZ is a square; or
the four points do not form a square.

A human-friendly version of the algorithm is as follows: given four distinct coplanar points, form the six sets of distances between the points. If four of these distances are the same and the remaining two distances are also equal but not the same as the first four, then the four points form a square with the two line segments with the second set (the pair) of equal lengths being the diagonals.

Also solved by Bard Crawford, Brian Leibowitz, Chester Claff, Christopher Leavitt, F. G. Bartlett, Harry Zaremba, Howard Stern, Joseph Keilin, Mary Lindenberg, Matthew Fountain, Michael Hennessey, Richard Heldenfels, Richard Hess, Richard Marks, Samuel Levitin, Steve Feldman, and Winslow Hartford.

N/D 2. For an office party, each person is supposed to bring a gift for someone else. The recipients are assigned to givers by writing each person's name on a slip of paper, putting the slips in a hat, and having everyone draw a slip. What is the probability that in an office of n people, no one draws his own name?

The proposer, Oren Cheyette, sent us the following solution, which he attributes to Veit Elser. Let Q_n denote the number of ways a successful (no match) draw can be performed by a group of n people. For low n, we have $Q_1 = 0$, $Q_2 = 1$, $Q_3 = 2$. The Q_n's satisfy the recursion relation

$Q_n = (n - 1)(Q_{n-1} + Q_{n-2})$

because there are two ways to get an n-permutation with no match from m < n permutations: take an n − 1 permutation with no match and exchange the n^{th} element with any of the n − 1, or take an n − 2 permutation and adjoin an interchanged pair. The desired probability is then $P_n = Q_n/n!$, so

$P_n = (n - 1)P_{n-1}/n + P_{n-2}/n = P_{n-1} - (P_{n-1} - P_{n-2})/n$,

or defining

$R_n = P_n - P_{n-1}$, we get

$R_n = R_{n-1}/n$ or

$R_n = (-1)^{n-1} R_1/n!$.

If we define P_0 to be 1, then $R_1 = -1$, and we find $R_n = (-1)^n/n!$. P_n is given the telescoping sum of R_k:

$$P_n = \sum_{k=1}^{n} R_k + P_0$$

or, finally, the desired probability is

$$P_n = \sum_{k=0}^{n} (-1)^n/n!$$

Harry Zaremba applied a sneaky inclusion-exclusion argument to produce a shorter solution: Since the events, "at least one person has selected his own name out of the hat," are not necessarily mutually exclusive, the probability of their occurrence is given by:

$$P = \binom{n}{1}(n-1)!/n! - \binom{n}{2}(n-2)!/n + \binom{n}{3}(n-3)!/n - \ldots + (-1)^{n+1}/n!,$$

or,

$$P = 1 - 1/2! + 1/3! - \ldots + (-1)^{n+1}/n!.$$

If Q is the probability of the event "no person draws his own name out of the hat," then,

$Q = 1 - P$ or

$$Q = 1 - [1 - 1/2! + 1/3! - \ldots + (-1)^{n+1}/n!] = \sum_{0}^{n} (-1)^i/i!$$

when the number of people n equals 6 or more, then the probability Q for all practical purposes equals e^{-1} or 0.368. Thus, Q is esentially independent of n for n > 6.

Mr. Cheyette reports that he has recently seen a variant of this problem in Polya's books on problems in analysis, where the solution was credited to Euler. Richard Hess notes that this problem is discussed in Ball's *Mathematical Recreations and Essays*, where it is traced to DeMontmorts in 1713.

Also solved by Avi Ornstein, Bin Ly, Edwin McMillan, Gerald Leibowitz, Harry Zaremba, Joseph Keilin, Mark Clements, Mary Lindenberg, Matthew Fountain, Michael Tamada, Pierre Heftler, Richard Hess, Roger Milkman, Steve Feldman, T.J.H. (no name was given), Raymond Gaillard, and Winslow Hartford.

N/D 3. Given a square matrix A (not necessarily invertible) satisfying AA = AA', where the prime signifies transpose operator, show that A = A' using matrix operation only, i.e without using normed algebras and approximating A by an invertible matrix.

The proposer Howard Stern offers the following: We make use of the fact that if BB' = 0 then B = 0. That is easy to prove using the trace operator of a matrix.

Let D = A − A'. It is sufficient to prove that DD' = 0.

DD' = (A − A')(A' − A) = AA' − AA − A'A' + A'A
= 0 − AA' + A'A (using the given and that A'A' = AA')
= A'A − AA.

Let B = DD' = A'A − AA'. Now it is sufficient to prove BB' = 0.

BB' = (A'A − AA')(A'A − AA')
= A'AA' A − A'AAA' − AA'A'A + AA'AA'
= A'AAA − A'AAA − AAA'A + AAAA (using the given)
= 0 − AAAA + AAAA (using the given)
= 0.

Therefore, A = A'. QED.

Also solved by Michael Tamada and Richard Hess.

N/D 4. What is the lowest number of current U.S. coins (1 cent through $1) for which there is no combination of coins that will equal in value a single coin? How many such quantities are there under 100?

The following solution is from Phelps Meaker, using the conventions P = penny, N = nickel, D = dime, Q = quarter, H = half-dollar, and $ = dollar:

1 P
2 2N
3 N+2D
4 N+2D+Q
5 3N+D+Q
6 5N+Q
7 5P+2D
8 6N+2D
9 8N+D
10 10N
11 2N+9D
12 4N+8D
13 3N+10P
14 10P+4D
15 10P+3N+Q+H
16 15N+Q
17 14N+3D
18 16N+2D
19 10P+8N+H
20 20N
21 20P+N
22 10P+9N+2D+Q
23 10P+8N+5D
24 5P+19N
25 25P
26 20P+6N
27 15P+10N+D+Q
28 25P+3Q
29 25P+3N+D
30 25P+5N
31 25P+5N+H
32 25P+5N+2Q
33 25P+N+7D
34 30P+4N
35 20P+14N+D
36 20P+16N
37 30P+7D
38 35P+3N
39 35P+3N+H
40 35P+3N+2Q
41 40P+D
42 40P+2N
43 40P+2N+H
44 40P+2N+2Q
45 35P+7N+3D
46 40P+6D
47 45P+N+H
48 40P+7N+Q
49 45P+3D+Q
50 50P
51 50P+H
52 50P+2Q
53 45P+5N+3D
54 45P+7N+2D
55 50P+5D
56 50P+2N+4D
57 50P+4N+3D
58 50P+6N+2D
59 50P+8N+D
60 50P+10N

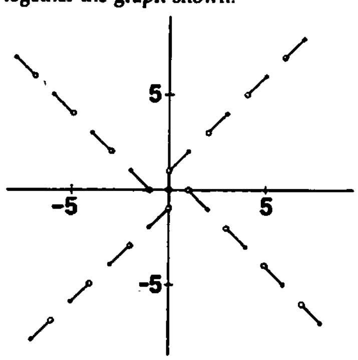
61	55P+3N+3D	81	-
62	55P+5N+2D	82	80P+2D
63	55P+7N+D	83	80P+2N+D
64	55P+9N	84	80P+N
65	60P+2N+3D	85	-
66	60P+4N+2D	86	-
67	60P+6N+D	87	85P+N+D
68	60P+8N	88	85P+3N
69	65P+N+3D	89	-
70	65P+3N+2D	90	-
71	65P+5N+D	91	90P+D
72	65P+7N	92	90P+2N
73	70P+3D	93	-
74	70P+2N+2D	94	-
75	70P+4N+D	95	-
76	70P+6N	96	95P+N
77	-	97	-
78	75P+N+2D	98	-
79	75P+3N+D	99	-
80	75P+5N	100	100P

The lowest number is 77, and there are 12 such numbers under 100.

Also solved by Naomi Markovitz, Avi Ornstein, Bill Wold, Brian Leibowitz, Donald Savage, Harry Zaremba, Matthew Fountain, P. Michael Jung, Winslow Hartford, and the proposor, Walter S. Cluett.

N/D 5. Find functions f and g satisfying
$f[f(x)] = x$
$g[g(x)] = -x$
for all real values of x.

Our final solution is from Charles Sutton:
Two simple solutions to the functional equation $f[f(x)] = x$ are $f(x) = -x$ and $f(x) = 1/x$, while the more general $f(x) = (a + bx)/(-b + cx)$ includes the first two as special cases. To find solutions of the equation $g[g(x)] = -x$ is more difficult. If $g(a) = b$, then $g(b) = -a$, $g(-a) = -b$, and $g(-b) = a$, so the four points (a,b), $(b,-a)$, $(-a,-b)$ and $(-b,a)$, one in each quadrant, all lie on the graph of $y = g(x)$. Since these points form the vertices of a square with center at the origin, it follows that the graph will be unchanged in form if rotated 90° about the origin, with four congruent parts, one in each quadrant. Since there can be only one value of y for each value of x, it is clear that the graph must consist of isolated pieces of curves, alternating between the upper and lower quadrants. Using straight-line segments of slope one in the first quadrant, one can piece together the graph shown.



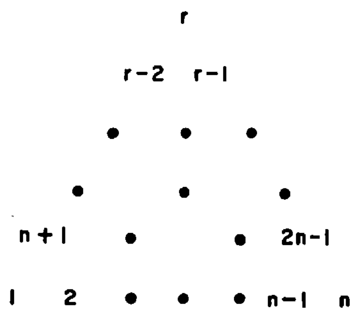
Small circles at the end of segments mean that the end point is missing, and the dot at the origin means that $g(0) = 0$. Infinitely many other solutions of the functional equation $g[g(x)] = -x$ can be constructed by the following procedure. Start with a set of isolated arcs of curves, all having positive slopes and all lying in the first quadrant above the line $y = x$, and construct a second set by reflecting the first set in the line $y = x$. Then adjust the two sets in such a way that the projections of these two sets on the x-axis completely fill the positive x-axis, with no points on the x-axis being the projections of points from both sets. [This will require a "missing point" at the end of each arc.] Then reflect the second set in the x-axis to obtain a third set. The first and third sets constitute the graph of the function for positive x. This is because the point (a,b) reflected in the line $y = x$ becomes (b,a), which becomes $(b,-a)$ when reflected in the x-axis. Rotate 180° and add the isolated point at the origin to obtain the rest of the graph.

Also solved by Avi Ornstein, Bin Ly, Brian Leibowitz, Gerald Leibowitz, Greg Huber, James Landau, Jerry Grossman, John Prussing, Joseph Keilin, Matthew Fountain, Michael Hennessey, Michael Tamada, P. Michael Jung, Pierre Heftler, Richard Hess, Robert Cherry, Winslow Hartford, and the proposor, Ronald Raines.

Better Late Than Never

JUL 1. Ray Kinsley was able to obtain the answer $(10^{10} + 3)$ on a programmable calculator unable to record numbers this large.

JUL 3. Stephen McAdam offers the following clarification:
If an n-triangle can be covered by 3 lines (with the parallel restriction), then 9 divides $n(n + 1)/2$, (so 9 divides either n or n + 1) and the number of lines in any direction is divisible by 3. First note that if there are k 3 lines involved, then there must be 3k points, so 3 divides $n(n + 1)/2$. We now number these points as follows:



We partition the set of lines into three subsets, A, B, and C, consisting respectively of those lines with positive, negative, and zero slope. Let the sizes of these sets be, respectively, a, b, and c. If L is one of the 3 lines, by s(L) we will mean the sum of the numbers of the three points which L covers. Note that if $L \in C$, then $s(L) \equiv 0 \bmod 3$, since s(L) is the sum of three consecutive integers. Also it is easily seen that if $L \in A$, then s(L) is the sum of three numbers of the form $x, x+y, x+y+(y-1)$(for some x and y), and so $s(L) \equiv 2 \bmod 3$. Similarly, if $L \in B$, then $s(L) \equiv 2 \bmod 3$.
Summing over all of our lines, we get $\Sigma s(L) = 1 + 2 + 3 + \ldots + r = r(r+1)/2$. We already have $r \equiv 0 \bmod 3$, and so $\Sigma s(L) \equiv 0 \bmod 3$. Now since $L \in C$ implies $s(L) \equiv 0 \bmod 3$, we see that $(\Sigma_A s(L)) + (\Sigma_B s(L)) \equiv 0 \bmod 3$, where the subscripts mean sum over the L in that set. Recalling that the size of A is a, since $L \in A$ implies $s(L) \equiv 2 \bmod 3$, $(\Sigma_A s(L)) \equiv 2a \bmod 3$. Similarly, $(\Sigma_B s(L)) \equiv 2b \bmod 3$. Therefore, $2a + 2b \equiv 0 \bmod 3$. It follows that $a \equiv -b \bmod 3$.
The above argument can be applied symmetrically. That is, choose some other side to be the "base" of the triangle so that (for instance) A becomes the set of lines with zero slope. Then we get that $c \equiv -b \bmod 3$. Already having $a \equiv -b \bmod 3$, we get $c \equiv a \bmod 3$. Again by symmetry, we conclude that in fact $a \equiv b \equiv c \equiv 0 \bmod 3$. Now from $c \equiv -b \bmod 3$, we get $c \equiv -c \bmod 3$, so that $a \equiv b \equiv c \equiv 0 \bmod 3$. This proves that the number of lines in any one direction is divisible by 3. Clearly the total number of lines is a + b + c which is divisible by 3, and so the total number of points, 3(a + b + c), is divisible by 9, as desired.

Proposers' Solutions to Speed Problems

SD 1. 1.714 seconds. He is on top of the image when he is directly under the lamp. [A regular problem based on this idea will appear next issue—ed.]

SD 2. The number of LED's that are lit in an LED display of the digits 1 through 9: