ALLAN J. GOTTLIEB

Fun with Trig

Since it has been over a year since I reviewed the criteria used to select solutions for publication, let me do so now.

As responses to problems arrive, they are simply put together in neat piles, with no regard to their date of arrival or postmark. When it is time for me to write the column in which solutions are to appear, I first weed out erroneous and illegible responses. For difficult problems, this may be enough; the most publishable solution becomes obvious. Usually, however, many responses still remain. I next try to select a solution that supplies an appropriate amount of detail and that includes a minimal number of characters that are hard to set in type. A particularly elegant solution is, of course, preferred. I favor contributions from correspondents whose solutions have not previously appeared, as well as solutions that are neatly written or typed, since the latter produce fewer typesetting errors.

Finally, let me credit David Griesedieck as the solver of **OCT 3**. Somehow his name disappeared when the column travelled from New York to Cambridge.

Problems

M/J 1. We begin with a two-part bridge problem from Doug Van Patter based on the following deal:

North:
♠ A 8 4
♥ A K 10
♦ A 7 6 5 4
♣ K J

West:
♠ K
♥ Q J 9 4 3 2
♦ Q 9 8 3 2
♣ Q

East:
♠ Q J 10 3 2
♥ 8 6 5
♦ J
♣ 10 7 5 4

South:
♠ 9 7 6 5
♥ 7
♦ K 10
♣ A 9 8 6 3 2

How does South make five clubs after an opening lead of the ♦Q, and what lead sets this contract?

M/J 2. As I look out my kitchen window and see our lake frozen solid, I am warmed by Phelps Meaker's N-sided flower garden described in the following problem, entitled "Fun with Trig":

A large bed of flowers and greenery is laid out in the form of a regular polygon of N sides. A walk composed of N trapezoidal concrete slabs surrounds the flower bed. A circumscribing circle passing through the outer corners of the walk and an inscribing circle tangent to the inner flats of the trapezoids have circumferences in the ratio of almost exactly 191:165. The total area within the outer periphery of the walk and the area of the walk itself are in the ratio of 4:1. Find N.

M/J 3. Lester Steffens wants you to explain a parlor trick he is fond of:

Given a number N between 500 and 1000, rapidly construct a series of numbers using eight of the nine digits from 1 through 9 once, the other digit twice, and a few zeros so that the series totals N. Example: if N = 642, a solution is

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 10
 20
334
 50
 60
 78
 90
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642
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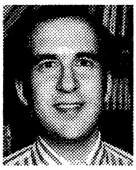
M/J 4. Winslow Hartford asks a multi-faceted problem:

As we generate geometric figures to represent $y = x^n$, we have "elements" consisting of points, lines, faces, cubes, etc. as n increases. For the number of points in each figure, we have (for $n > 0$) $P = 2^n$. Derive the number of lines and faces in a five-dimensional hypercube.

M/J 5. Eric Schonblom tells us about his 8.01 doodles, with an apology for tardiness: "Having discovered this during



SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, ASSOCIATE RESEARCH PROFESSOR AT THE COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y., 10012.

a physics lecture over 30 years ago, I'm a little slow in sharing it. It's a paper-and-pencil puzzle but is most easily stated in terms of scrabble tiles":

Take the letters in the first half of the alphabet in order. Place the A on the table. Place the B next to one of the four sides of the A. Place the C next to the one of the six sides at the AB (or BA) pair. Then add the D and so on. If you do this correctly, when you reach the letter M your will have created a crossword-puzzle matrix of complete common English words, no proper names, no foreign words, and no acronyms or abbreviations. Having solved the problem as posed, can you add one or more letters to the A-M set and still retain complete words?

Speed Department

SD 1. Jerry Grossman wants to know what is so interesting about
$f(x) = x^{1/\log x}$,
where log is the natural logarithm.

SD 2. David Evans designed a single elimination tennis tournament for 37 contestants to have the minimum number of byes. How many matches were played?

Solutions

JAN 1. South is on lead and is to take six tricks against best defense with hearts as trump:

North:
♠ 9 8 3 2
♥ 10
♦ A Q
♣ —

West:
♠ J
♥ 8
♦ J 10 9 8
♣ 5

East:
♠ 10
♥ 6
♦ K 6
♣ K J 6

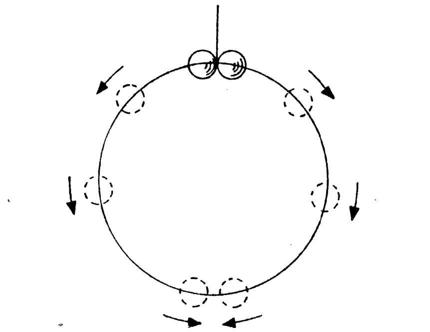
South:
♠ A K 7
♥ —
♦ —
♣ A Q 4 3

Ben Feinswog solved the problem by losing the ♠7 and discarding the ♠A and ♠K:
South plays the ♣A (removing West's exit), discarding the ♦Q from dummy, and then leads the ♠7 to West's ♠J. On the forced red-card return, South plays dummy's winning ♥10, and ♦A, discarding the ♠A and ♠K from hand, and claims the balance with dummy's last three spades.

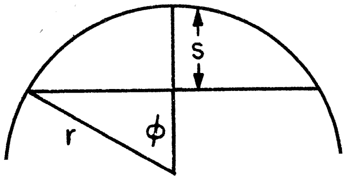
Also solved by Avi Ornstein, Doug Van Patter, Edgar Rose, Ellen Kranzer, Joe Hahn, John Lacy, Larry Wischhoefer, Matthew Fountain, Red Clevenger, Robert Lax, Roy Schweiker, Tim Maloney, Walter Cluett, Winslow Hartford, Jim Landau, John Rule, Richard Hess and Emmet J. Duffy.

JAN 2. A smooth, rigid, and circular hoop hangs from a rigid support by an ideal, extensionless string. Two small beads slide along the hoop (like beads of a necklace) with negligible drag and friction. The beads are slid to the top of the hoop and released. How massive must each bead by to spontaneously lift the hoop?

Matthew Fountain was pleased to submit a solution but even more pleased to report that his wife has responded perfectly to a recent cataract operation:
The two beads must total three times the mass of the hoop. As each bead slides outward, its circular motion causes a centrifugal force with an upward



component. At the same time its downward acceleration increases, decreasing the force that it exerts upon the hoop. The maximum lifting force occurs when the sum of these two effects is greatest. Until the hoop actually moves, a bead does no work. Therefore, its gain in kinetic energy equals its loss in potential energy. Thus, $(1/2)mv^2 = smg$, where s = vertical drop, m = mass, g = gravitational constant, and v = velocity.



When the bead has traversed an arc ϕ on the hoop of radius r, the vertical drop
$s = r(1 - \cos\phi)$.
The centrifugal force is mv^2/r, with an upward component
$(mv^2/r)\cos\phi = 2smg(\cos\phi)/r = 2mg(1-\cos\phi)\cos\phi$. Gravity produces a force mg acting downward through the bead. When the bead has traversed through the arc ϕ, the component of this force toward the center of the hoop is $mg(\cos\phi)$. In turn, the downward component of the radial component is $mg(\cos^2\phi)$. The lifting force F exerted by a bead is
$F = 2mg(1 - \cos\phi)\cos\phi - mg(\cos^2\phi) = 2mg(\cos\phi) - 3mg(\cos^2\phi)$.
The maximum and minimum values of F occur when
$dF/d\phi = -2mg(\sin\phi) + 6mg(\cos\phi)(\sin\phi) = 0$.
Thus the minimum lift occurs when $\sin\phi = 0$ and the maximum lift occurs when $\cos\phi = 1/3$:
$F_{max} = (1/3)2mg - (1/3)^2 3mg = (1/3)mg$.
Each bead will lift up to one-third of its own weight.

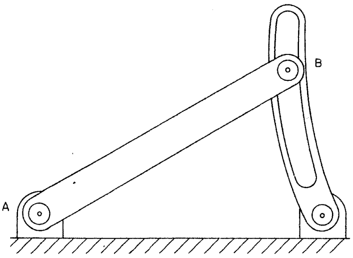
Also solved by David Smith, Gary Heiligman, Harry Garber, Harry Zaremba, John Lacy, John Prussing, Ken Haruta, Jim Landau, Richard Hess, Peter Kramer, and the proposer, Bruce Calder.

JAN 3. A man received a check calling for a certain amount of money in dollars and cents. When he went to cash the check, the teller made a mistake and paid him the amount which was written in cents in dollars, and vice-versa. Later, after spending $3.50, the man suddenly realized that he had twice the amount of money the checked called for. What was the amount on the check.

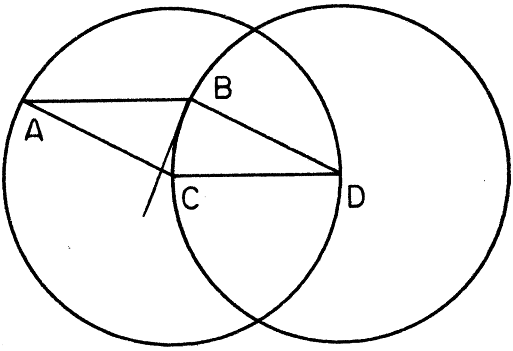
Red Clevenger provides a "down home" solution:
The problem reminds me of my eighth-grade teacher in rural Afton, Okla., who had several dollar-and-cents algebraic problems which she always wanted us to solve by using one variable. I preferred using two variables in simultaneous equations. She did drive home the lesson, however, of 100¢ = 1$.
Let
$ = the dollar amount of the check, and
c = the cents amount;
then the problem stated algebraically is:
$100c + \$ - 350 = 2(100\$ + c)$. (1)
The other equation results from the difference between the number of cents received ($) and twice the number of cents on the check (c) which must be − 50 since c is greater than $. Stated algebraically:
$\$ - 2c = -50$. (2)
Solving (1) and (2) results in $ = 14 and c = 32, and thus the check amount was $14.32.

Also solved by Winslow Hartford, Allan Benson, Anthony Lombardo, Avi Ornstein, Charles Sutton, David Smith, Edgar Rose, Frank Carbin, Fred Steigman, Gary Drlik, George Byrd, George Aronson, Harry Garber, Harry Zaremba, Howard Stern, James Michelman, John Lacy, John Prussing, Ken Haruta, Larry Wischhoefer, Leon Tabak, Marion Berger, Matthew Fountain, Michael Jung, Naomi Markovitz, Norman Spencer, Peter Silverberg, Phelps Meaker, Ronald Martin, Steve Feldman, Ted Numata, Thomas Stowe, Everett Leroy, Tim Maloney, Jim Landau, Ronald Raines, Richard Hess, George Parks, and the proposer John Rule.

JAN 4. A rigid arm pivots around the fixed point A. At the end of the arm is a follower (B) which runs in a curved track. The track pivots about the fixed point C. If AB = AC = r, find the shape of the track such that its slope at C is always vertical.



The problem as stated was rather easy. Surprisingly, several readers noticed that the diagram indicated a vertical slope at B, not C. This was indeed more interesting. Charles Sutton writes:
The problem as stated is trivial, since clearly a track in the shape of a semi-circle of radius r with center at A would remain stationary and its slope at C would always be vertical. I assume what was intended was that the slope of the track at B should always be vertical. This really had me going for a while. I set it up in rectangular coordinates, using anlaytic geometry and calculus, and ended up with an impossible differential equation. Then I tried polar coordinates, used a few trig identities and calculus, and got a real simple differential equation that integrated to give me a *circle*. Once you know what you're looking for, you need only elementary geometry. For the slope at B to remain vertical, the shape of the curved track is determined by the facts that AC = AB = r and the tangent to the track at B is perpendicular to the line at AC. Imagine that the track is held fixed; then point A will have to move on a circle of radius r. The accompanying diagram shows that the track is an arc of a circle of radius r with center at D. The tangent to the circle at B is perpendicular to both BD and AC, so AC must be parallel to BD. Also AC = BD = r, so ABCD is a parallelogram and AB = CD = r.



Also solved by David Smith, Gary Heiligman, George Byrd, Harry Garber, Harry Zaremba, Howard Stern, John Lacy, Jordan Wouk, Matthew Fountain, Phelps Meaker, Red Clevenger, Winslow Hartford, Richard Hess, Peter Kramer, and the proposer, Floyd Kosch.

Better Late Than Never

Y1984 Randall Whitman, William Thompson, Rick Lufkin, Marion Berger, Al Weiss, Matthew Fountain, Donald Trumpler, and Alan Katzenstein each improved on the published solution. When combined, their efforts yield the following list of revisions:

$2 = 1^9 \times 8/4$	$26 = (1 + 9/4) \times 8$
$3 = 4 - 1^{98}$	$31 = 49 - 18$
$4 = 1^{98} \times 4$	$37 = 1 + 9(8 - 4)$
$5 = 1^{98} + 4$	$41 = 49 - 8 \times 1$
$6 = (49 - 1)/8$	$42 = (49 - 8) + 1$
$7 = 91 - 84$	$49 = 49 \times 1^8$
$8 = 8 \times 1^{94}$	$68 = 81 - (4 + 9)$
$10 = 1^{48} + 9$	$70 = (9 - 1/4) \times 8$
$14 = 14 \times (9 - 8)$	$76 = 94 - 18$
$16 = (1 + 9 - 8)4$	$81 = 4 \times 18 + 9$
$18 = 9 \times 4 - 18$	$99 = 98 + 1^4$
$22 = (89 - 1)/4$	$100 = (1 + 9)^{8/4}$

JUL 2. Howard Stern notes that this problem appears in Martin Gardner's *Mathematical Circus*.

OCT 3. John Stackpole notes that the very week his **F/M 1** issue of *Technology Review* arrived, the Maryland Lotto game had ten participants match all six numbers and 827 match five out of six, giving strength to Griesedieck's assertion that the selection of numbers by participants is not random. Jonathan Hardis, the proposer, remarks that never have any of the large lotteries had a positive expected return for the player and also expresses the oft-stated political belief that these games constitute a regressive tax since they played to a disproportionately large extent by the poor.

Proposers' Solutions to Speed Problems

SD 1. It is constant, i.e. independent of x.

SD 2. 36. One contestant is eliminated each match.