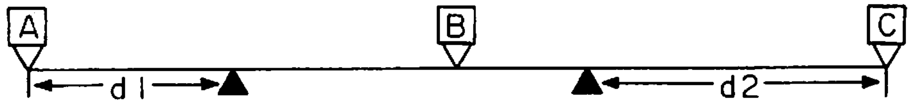
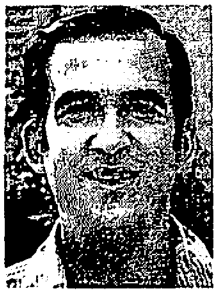


Why Is Beer So Cheap in Mexico?



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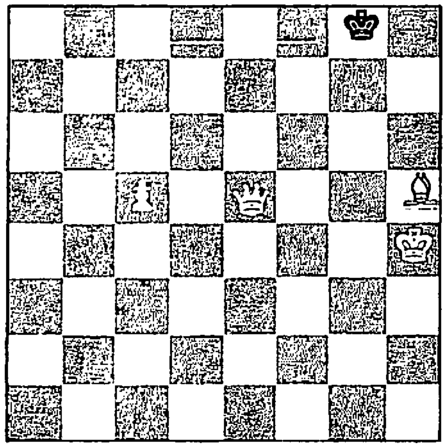
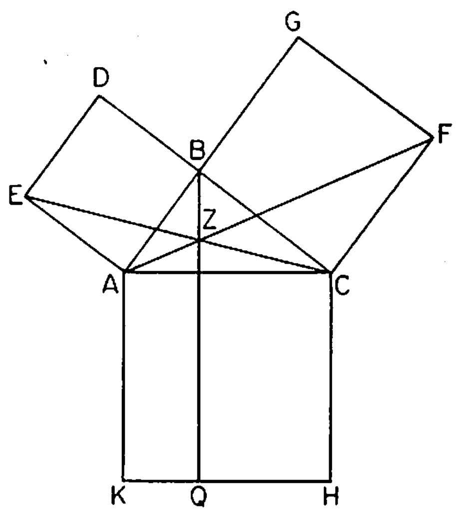
Two readers have commented on remarks I made in previous issues: In July I reported a story concerning Winthrop Leeds' study of Pythagorean triples. J. Meier writes that he spent many enjoyable childhood mornings doing mental arithmetic, often with Pythagorean triples. A few years ago I noted that my former Baker House softball teammate John Rudy used unknowingly to boost my ego by complaining that my throws from deep short would hurt his hand at first base. Now John writes of a more recent episode: "We were playing a softball game and somehow I was chosen to play first (they didn't know about Baker House [obviously — ed.]). I have a 19-year-old girl working for me who throws as hard as you did; last year our first baseman got a broken finger as a result of a hard throw by her. I survived."

Problems

N/D 1 This month's chess problem is from Bob Kimble: given the situation shown below, White is to mate in three moves:



N/D 2 A truly classic problem from David Lukeris. In Euclid's proof of the Pythagorean theorem, the figure shown below is drawn. ABC is a right triangle; B is a right angle; ABDE, BCFG, and ACHK are the squares on the sides; and BQ is parallel to AK. The problem is to determine if the three lines EC, BQ, and AF meet at a point Z as they appear to in the drawing below.



N/D 3 Frank Rubin has a massless beam of length L supported by two stanchions at distances d1 and d2 from the ends. The beam is loaded with point masses A, B, and C at the ends and midpoint as shown at the top of this page. What is the downward force on each stanchion?

N/D 4 Alan Davis has a clock with hour, minute, and second hands initially at the 12:00 position. Will the three hands ever divide the clockface into three equal parts? If so, when?

N/D 5 Our final regular problem is from Harry Zaremba. Six different numbers are selected arbitrarily from eight positive consecutive integers. The resulting selection includes the smallest and largest of the eight numbers and can be separated into three pairs of numbers, each of which contains consecutive numbers. The sum of the six integers is three times a number N, and the sum of their cubes equals the cube of N. Find N and each of the integers selected.

Speed Department

SD 1 Steve Chilton has a question about this story. At a town on the Mexican-American border during the last century a very peculiar monetary situation arose. In Mexico a U.S. dollar was worth 90 Mexican cents, while in the US a Mexican dollar was worth 90 U.S. cents. One morning a cowboy stopped in a U.S. bar and bought a 10-cent (U.S.) beer. He paid a U.S. dollar and received a Mexican dollar in return. He then went to a Mexican bar and bought a 10-cent (Mexican) beer. He paid with the Mexican dollar he previously received and received a U.S. dollar in return. After a full day of drinking he returned home with a U.S. dollar and a hangover. Who paid for the beer?

SD 2 A geometry quicky from L. Steffens. Given eight equal cubes assembled to form a $2 \times 2 \times 2$ cube, how many paths are there from a small cube to the one diagonally opposite, defining a path as a set of cubes through which a straight line can be drawn?

Solutions

JUL 1 You wish to maximize your chances in playing this nine-card suit:
Dummy: A K 10 x
Declarer: x x x x x
You play the A from the Dummy, and an honor falls on your right. It is well known in bridge circles that the odds are now nearly 2:1 in favor of finessing West for the other honor (Rule of Restricted Choice), provided that East is known to play either the Q or J at random from a doubleton Q-J holding. The problem is to consider what happens as the combined holding shrinks from nine cards to five while still maintaining the two honors (the limiting case is A K 10 in Dummy) and calculate the exact odds for the finesse in each case (nine cards, eight, etc.).

Matthew Fountain sent us the following solution: When Declarer and Dummy combined hold N cards of the suit, the odds in favor of finessing West are:

N	9	8	7	6	5
Odds	11/6	5/3	3/2	4/3	7/6

The "fall" of the honor implies a forced play. East started holding either one or two honors and no other cards of the suit. East and West combined started with $13 - N$ cards of the suit, and $26 - (13 - N) = 13 + N$ cards of other suits. The finesse against West succeeds when East has started with a singleton. The odds in favor of finessing West are (number of East hands with the singleton honor)/(number of East hands with the doubleton honor). The number of possible East hands with singleton honor is (number of ways of taking 1 honor from 2 honors)(number of ways of taking 12 cards from $13 + N$ cards) $= 2(13 + N)!/12!(1 + N)!$. The number of possible East hands with doubleton honors is (number of ways of taking 2 honors from 2 honors) (number of ways of taking 11 cards from $13 + N$ cards) $= 1(13 + N)!/11!(2 + N)!$. The odds in favor of finessing West are $2(13 + N)!/12!(1 + N)!/(13 + N)!/11!(2 + N)! = (2 + N)/6$.

The proposer, Douglas Van Patter, believes that for $N = 9$ the correct answer is 15:6.

JUL 2 A rocket containing 60 gallons of fuel approaches the moon (gravity is 5 ft./sec./sec.) tail-first with an initial velocity of 50 ft./sec. when at an altitude of 500 ft. Each second thereafter an integral number of gallons of fuel may be burnt, causing an upward acceleration of x ft./sec./sec., where $x = 2f - 5$. Find the sequence of one-second fuel burns that maximizes the maximum f while yielding a soft landing.

Harry Zaremba responded with a fine analysis. The maximum rate at which fuel can be burned to achieve a soft landing at zero velocity is $f = 30$ gal./sec. The descent would be composed of four stages of motion—two of free fall alternated with two one-second engine burns. The final velocity and distance travelled at the end of each stage of motion are presented in the analysis below. Positive values of distance, velocity, and acceleration are assumed to be in the direction toward the moon's surface, and the expression for x is presumed to be the net upward acceleration during an engine burn.

Free fall—first stage:

$v_1 = 50 + 5t_1$

$d_1 = 50t_1 + 5/2(t_1^2)$

Engine burn—second stage:

$v_2 = (50 + 5t_1) - (2f - 5)t_2 = 5t_1 - 5$

in which $f = 30$ and $t_2 = 1$ sec.

$d_2 = (50 + 5t_1)t_2 - ½(2f - 5)t_2^2 = 5t_1 + 22.5$

Free fall—third stage:

To assure a zero velocity at the end of the fourth stage for a soft landing, the final velocity at the end of the third stage must equal $(2f - 5)$, or $v_3 = 2 \cdot 30 - 5 = 55$ ft./sec. Thus,

$v_3 = 55 = (5t_1 - 5) + 5t_3$,

from which $t_1 + t_3 = 12$. (1)

$d_3 = (5t_1 - 5)t_3 + 5/2(t_3^2)$.

Engine burn—fourth stage:

$v_4 = 55 - (2f - 5)t_4 = 55 - (60 - 5) = 0$, where $t_4 = 1$ sec.

$d_4 = 55t_4 - ½(2f - 5)t_4^2 = 55 - 27.5 = 27.5$ ft.

After equating the sum of distances d_1 through d_4 to the initial altitude of 500 ft. and substituting $t_3 = 12 - t_1$ from equation (1) in the third stage, the time duration t_1 for the first stage will be $t_1 = 2.5$ sec. Therefore, $t_3 = 9.5$ sec., and the total time of descent will be

$T = t_1 + t_3 + 2 = 14$ sec.

The distances travelled during each stage will be $d_1 = 140.625$ ft.; $d_2 = 35$ ft.; $d_3 = 296.875$ ft.; and $d_4 = 27.5$ ft.

Also solved by Matthew Fountain and the proposer, Roy Sinclair.

JUL 3 Watchmakers all over the world display a symmetrical watch face in which the hands are set at 8:18. The problem is to test the idea that this watchface division goes back to the "Golden Mean" of the Greeks, in which the most esthetically pleasing division of space was believed to be that in which the larger part occupied $[(\sqrt{5} - 1)/2]$, or approximately 72 percent, of the total area. Is there an arrangement of the hands near 8:18 that is both symmetrical and in accord with the Golden Mean? If not, what is the time at the nearest symmetrical point and at the Golden Mean point?

John Prussing submitted the following solution. There is no arrangement of the hands that is both symmetrical about the vertical axis and divides the clock face exactly according to the golden mean. It is easy to show, using the fact that the hour hand moves at a constant rate of 30°/hr. and the minute hands moves at 360°/hr that there are 12 times at which the hands are symmetrical. The times (in units of hours past 12:00) are given by 12n/13, where n = 1, 2, . . ., 12. Of these times 8:18:28 and 3:41:32 come closest to dividing the clock face according to the golden mean

$G = (\sqrt{5} - 1)/2 \sim 0.618.$

For both of these times the fractional area bounded by the hands is 0.615. There are 22 unique arrangements of the hands that divide the clock face according to the Golden Mean. These times are given by 12(n − G)/11 and 12(n + G − 1)/11,

where n = 1, 2, . . ., 11. Of these, the times closest to the times of symmetry are 8:18:38 and 3:41:22. These differ from the symmetry times by only 10 seconds!

Also solved by Michael Jung, Winslow Hartford, Emmet Duffy, David Evans, Bruce Garetz, Frank Carbin, Steve Feldman, Dennis Sandow, Harry Zaremba, and Matthew Fountain. Several of the responders noted that the area is 62 percent, not 72 percent.

JUL 4 Find a rational number (other than 41/12) such that its square, when increased by 5, remains a square.

Judith Longyear notes that whenever

$(a/b)^2 - N = (c/b)^2$, and

$(a/b)^2 + N = (d/b)^2$, then for

$a' = Nb^2d^2 + a^2c^2$

$b' = 2abcd$

$c' = Nb^2d^2 - a^2c^2 = 2Na^2b^2 - c^2d^2$

$d' = 2Na^2b^2 + c^2d^2$, we also have

$(a'/b')^2 - N = (c'/b')^2$, and

$(a'/b')^2 + N = (d'/b')^2$.

Thus from one solution infinitely many others follow. Professor Longyear then proceeds to show that these are the only solutions. Her proof is available from the editor.

Daniel Grayson recognized that the equations

$a^2 + 5 = b^2 = c^2 - 5$

describe an elliptic curve. By applying techniques from elliptic function theory, he was also able to show that infinitely many solutions occur (details available from the editor).

John Wrench included with his solution several references to the literature. Apparently this problem was considered and partially solved by Fibonacci more than 760 years ago. Mr. Wrench recommends that interested readers consult Uspensky and Heaslet, *Elementary Number Theory*, p. 427; Dickson, *History of the Theory of Numbers*, pp. 459-472; and Sierpinski, *Elementary Theory of Numbers*, pp. 63-67. The simplest solution after 41/12 is 3344161/1494696.

Also solved by Matthew Fountain, John Prussing, Charles Sutton, David Evans, Winslow Hartford, and the proposer, Smith Turner.

JUL 5 Suppose that some new type of photographers' light bulbs undergoing a life-test for one week burned out as follows:

Sunday: ½ of the bulbs + ½ of a bulb,
Monday: ⅓ of the bulbs left + ⅓ of a bulb,
Tuesday: ¼ of the bulbs left + ¼ of a bulb, and so on progressively until
Saturday: ⅛ of the bulbs left + ⅛ of a bulb.

Assuming that there is only one filament in each bulb, which is the least number of bulbs that could have been left when the test ended? If the fractions had progressed in reverse order (starting with ⅛ of the bulbs + ⅛ of a bulb on Sunday), would the final result have been the same? Why?

The following solution is from Emmet Duffy.

Let the number of bulbs under test be A, where A and all subsequent letters B through G are integers. Then A/2 + 1/2 is the first subtraction.

Let A/2 + 1/2 = B; then

A = 2B − 1 (1)
less B − 1/2 + 1/2
B − 1
less B/3 − 1/3 + 1/3
Let B = 3C; then (2)
B − 1 = 3C − 1
less C − 1/3 + 1/3
2C − 1
less 2C/4 − 1/4 + 1/4
Let C = 2D, then (3)
2C − 1 = 4D − 1
less D − 1/4 + 1/4
3D − 1
less 3D/5 − 1/5 + 1/5
Let D = 5E; then (4)
3D − 1 = 15E − 1
less 3E − 1/5 + 1/5
12E − 1
less 2E − 1/6 + 1/6
10E − 1
less 10E/7 − 1/7 + 1/7
Let E = 7F, then (5)
10E − 1 = 70F − 1
less 10F − 1/7 + 1/7
60F − 1
less 8½F − 1/8 + 1/8
Let F = 2G, then (6)
60F − 1 = 120G − 1
less 15G − 1/8 + 1/8
105G − 1

Then from equations (6) through (1), A = 840G − 1. The minimum positive remainder occurs when G = 1 and is 104 with 839 at the start of the test, or 840 − 1. As 840 is the least common multiple of numbers 2 to 8, subtracting one half plus one half of a bulb is the same as multiplying 840 by 1/2 and then subtracting 1. The total subtraction will then result in:

840 (1/2 × 2/3 × 3/4 × 4/5 × 5/6 × 6/7 × 7/8) − 1 = (840 × 1/8) − 1 = 104.

If the order of subtraction is reversed the result will be:

840 (7/8 × 6/7 × 5/6 × 4/5 × 3/4 × 2/3 × 1/2) − 1 = 104.

Any order of subtraction will give the same remainder, 104, but some orders will not result in an integer for a subtraction. It is interesting to note that if the problem had only been the reversed order of subtraction, starting with 1/8 of the bulbs plus 1/8 of a bulb, the answer would be 7 bulbs under test and none remaining at the end of the test.

Also solved by Harry Zaremba, Michael Jung, Dennis Sandow, Bruce Garetz, Judith Longyear, Matthew Fountain, David Evans, Winslow Hartford, and John Prussing.

Better Late Than Never

1981 OCT 1 Harry Zaremba was able to obtain an exact solution (we previously had only upper and lower bounds—see the July issue). His detailed analysis, available from the editor, shows that the volume is 0.42215775 cubic units.

N/D 2 A. Singer has responded.

1982 FEB 2 A. Singer and Alan Feldman have responded.

M/A 1 John Rutherford has responded.

M/A 3 Irving Hopkins feels that no proof was given that the circumscribed polygon has the maximum area.

M/A 5 William McGuinness has responded.

M/J 1 Everett Leroy and A. Singer have responded.

M/J 4 A. Singer has responded.

JUL SD 1 David Evans and Stephen Kliment note that the answer is 9567 + 1085 = 10652.

JUL SD 2 John Prussing notes that the correct solution is u = 0.684746, v = 0.816541. The alternating sequence given in July is caused by the iteration algorithm used—namely successive approximations. The algorithm does not converge but oscillates between two values v1 and v2 that satisfy cos(v1) = arctan (v2) and cos(v2) = arctan(v1). An analogy is the linear problem u = v, v = 1 − u. The solution is u = v = 1/2, but a successive approximation algorithm will oscillate between any initial value v1 and the value v2 = 1 − v1.

Proposers' Solutions to Speed Problems

SD 1 The people whose money was devalued when they crossed the border with it. After all, how did those U.S. dollars get in the Mexican bar's till?

SD 2 13 paths.