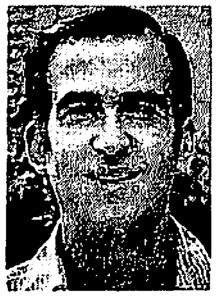
Puzzle Corner
Allan J. Gottlieb

Water In Your Ice Cream Cone



Allan J. Gottlieb, '67, is associate research professor of mathematical sciences at the Courant Institute of Mathematical Sciences, New York University; he studied mathematics at M.I.T. and Brandeis. Send problems, solutions, and comments to him at the Courant Institute, New York University, 251 Mercer Street, New York, N.Y., 10012.

Alice and I want to thank you all once again for kind words concerning our new son. Larry Marden notes a coincidence: "When I began reading your column, I couldn't help feeling that something sounded very familiar. You see, on Sunday evening, March 14, my wife also gave birth to our first child, also a boy, but 'only' weighing 8 lbs. 10½ oz. By the way, I completely agree with all your comments about the delivery process and the participants."

Special note, on a different subject: both chess and (especially) "speed" problems are in short supply.

Problems

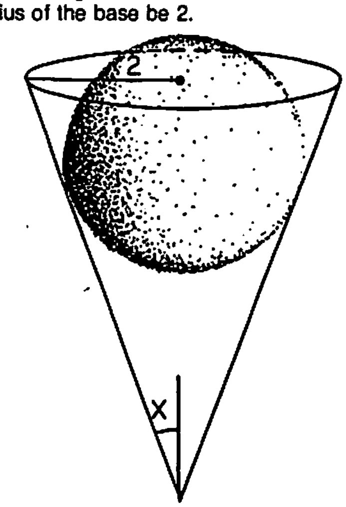
OCT 1 We begin with another of Emmet Duffy's seven-card bridge problems. South is on lead with hearts as trump and is to take all tricks against the best defense.

```
              ♠ A J
              ♥ 5
              ♦ K Q
              ♣ 10 8
♠ K 2                      ♠ 4 3
♥ —                        ♥ 6 3
♦ J 9 8                    ♦ 7
♣ 9 7                      ♣ 6 5
              ♠ Q
              ♥ 8 4
              ♦ A 10 3 2
              ♣ —
```

OCT 2 Stephen Spacil sent us the following set of cryptarithmetic puzzles created by Nobuyuki Yoshigahara. The set is entitled "Seven and Twelve:"

```
   THREE             EIGHT
   SEVEN             EIGHT
   SEVEN             SEVEN
 + SEVEN             SEVEN
 ---------           SEVEN
  TWELVE × 2       +ELEVEN
                   ---------
     ONE           TWELVE × 4
   SEVEN
   SEVEN              FIVE
   SEVEN              FIVE
   SEVEN             SEVEN
 + SEVEN             SEVEN
 ---------           SEVEN
  TWELVE × 3         SEVEN
                    ELEVEN
                   +ELEVEN
                   ---------
                   TWELVE × 5
```

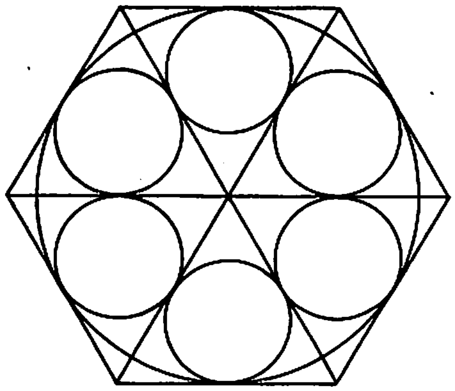
OCT 3 Edmund Nadler likes to fill his ice cream cones with water and spheres. He obviously has a lot to learn. Mr. Nadler writes: Given an ice cream cone filled with water, how large a sphere displaces the most water? Let the half angle of the cone be x, and let the radius of the base be 2.



OCT 4 Reino Hakala wants to know the closed form (not infinite series) solution of $dy/dx = x - y^2$.

OCT 5 Lou Anne Nesta asks:
Given a regular hexagon with an inscribed circle, what is the ratio of the area of the six

smaller circles (see drawing) to the area of the inscribed circle?



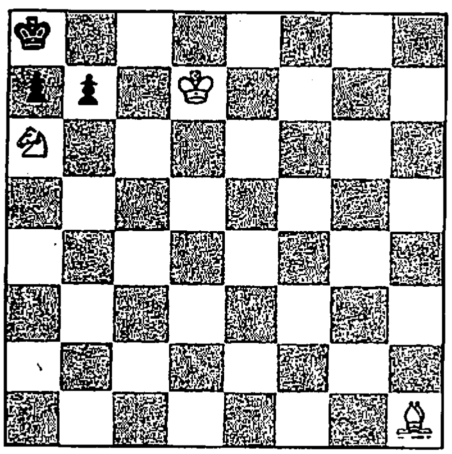
Speed Department

SD1 Lester Steffens suggests:
DAVID × (something) = BENDIX.

SD2 Emmet Duffy wants to know the last three digits of 7^{9999}.

Solutions

M/J 1 Given the positions shown, White is to play and mate in three moves. In the first case, the Black pawns move *up* the board, in the second the Black pawns move *down* the board.



Larry Marden had little trouble with either case in this bidirectional problem: Whether Black moves up or down the board, White must avoid the obvious pitfall of stalemate, to which most of the available moves lead. A little experimentation leads to the following:

Down the board:

White:	*Black:*
B—B6	P × B (forced)
K—B8	P—B4 (forced)
N—B7 (mate)	

Up the board:

White:	*Black:*
K—B6	P—N8 (N) (ch)
K—B7 (ch)	N—B3
B × N (mate)	
or, even easier:	
K—B6	P—N8 (R)
K—B7 (mate)	
or, finally:	
K—B6	P—N8 (Q or B)
N—B7 (ch)	(Q or B) × N (forced)
K × (Q or B) (mate)	

Also solved by Elliott Roberts, Ed Chang, Martin Wohl, Richard Hess, Raymond Gaillard, David Evans, Rony Adelsman, Michael Jung, Thomas Sico, Dave Simen, Matthew Fountain, and George Sharman.

M/J 2 How many regular decompositions are there of the sphere? (A regular decomposition is one in which the surface is subdivided into congruent spherical regular polygons, where a spherical polygon is called regular if all its vertex angles are equal.)

As Allen Tracht points out, the answer is either five or infinity: There are five regular decompositions based on the five regular solids known to Euclid plus an infinite number composed of lunes. These decompositions are tabulated as follows:

Number and type of spherical polygons		*Number of vertices*	*Vertex angles in degrees*
4	Three-sided	4	120
6	Four-sided	8	120
8	Three-sided	6	90
12	Five-sided	20	120
20	Three-sided	12	72
N	Two-sided	2	360/N

Also solved by Richard Hess, Matthew Fountain, and the proposer, Roy Sinclair.

M/J 3 Given an integer $n > 0$ and an odd integer m, prove that any odd integer has one and only one mth root mod 2^n.

Richard Hess begins by showing uniqueness. That is, assume

$2k + 1 \equiv q^m \bmod 2^n$, and

$2k + 1 \equiv p^m \bmod 2^n$.

Then both p and q must be odd, and by subtracting we get

$0 \equiv q^m - p^m \bmod 2^n$. But

$q^m - p^m = (q - p)(q^{m-1} + q^{m-2}p + \ldots + qp^{m-2} + p^{m-1})$,

and the second factor contains an odd number of terms each of which is odd. Thus, the second factor is odd and hence not zero mod 2^m. This shows that

$q \equiv p \bmod 2^m$,

so at most one mth root exists. To show that an mth root does exist, Mr. Hess counts: the 2^{n-1} odd numbers $1^m, 3^m, 5^m, \ldots (2^n - 1)^m$ are all distinct mod 2^m as shown above. Thus they must include all the odd numbers from 1 to $2^n - 1$.

Also solved by Dave Simen, Allen Tracht, John Prussing, and the proposer, Alan LaVergne.

M/J 4 Determine the 145 digits in this skeleton division in which each "x" represents one digit. The nine quotient digits shown in bold type form a repeating decimal (i.e., the group as a whole repeats indefinitely). Divisor and dividend have no common factor. Find the digits; there is only one possible answer.

```
xxxxxx   xxxxxxxxx.xxxxxxxxxx
           xxxxxx
           xxxxxxx
            xxxxxx
            xxxxxxx
            xxxxxxx
             xxxxxxx
              xxxxxx
              xxxxxxx
              xxxxxxx
               xxxxxxx
               xxxxxxx
                xxxxxxx
                xxxxxxx
                 xxxxxxx
                 xxxxxxx
                        xxxxxx
                        xxxxxx
                        xxxxxx
```

Matthew Fountain offers us a step-by-step derivation:

```
667334   7752341        11.6168830001     (a)
         667334    (b)    DEFGH  C
         1079001   (c)
          667334   (d)
          4116670   (e)
          4004004   (f)
           1126660   (g)
            667334   (h)
            4593260   (i)
            4004004   (j)
             5892560   (k)
             5338672   (l)
              5538880   (m)
              5338672   (n)
               2002080   (o)
               2002002   (p)
                   780000   (q)
                   667334   (r)
                   112666   (s)
```

The solution starts with recognition that line (q) begins with two digits, say AB, followed by four zeros, and that line (a) ends in three zeros followed by a non-zero, say C. Therefore (the divisor)(CDEFGH) = (AB)(10^{9}) − (AB) = (AB)(2997)(333667), where DEFGH is the series of digits starting the repeating decimal in line (a). C and D are smaller than the remaining digits in the series, as lines (r) and (h) are shorter than lines (j), (l), (n), and (p). As 333667 is a prime and neither it nor its double, 667334, fits the description of CDEFGH, it is a factor of the divisor. If 333667 is the divisor, lines (d) and (h) require that [line (c)]/333667 be less than 2.93, that is, of the form 2. × 2.. But line (c) is at least 1000000 and 1000000/333667 = 2.99+. Therefore 667334 is the divisor and (AB/2)(2997) = CDEFGH. Lines (r) and (h) now show C = D = 1. Lines (q), (r), and (s) show AB − 66 = xx so that AB exceeds 76. Substitution of AB = 78, 80, and 82 in (AB/2)(2997) yields 116883, 119880, and 122877, respectively. AB = 78 and CDEFGH = 116883 as AB = 80 yields H = 0 and all larger values for AB yield D larger than 1. Line (s) is 780000 − 667334 = 112666 so that line (g) is 1126660. With line (e) ending in 0 and line (g) ending in 60, line (f) must end in 4. Therefore the third digit of the quotient is 6. Lines (b) and (d) contain six digits, showing the first two digits of the quotient are ones. The dividend = (6667334)(11.6168830001..) = 7752341. The rest of the problem is just arithmetic.

Also solved by Edwin McMillan, Richard Hess, Norman Wickstrand, and Dave Simen.

M/J 5 Consider the problem of dividing a cake equally between two people A and B using only a knife. Assuming A and B always try to maximize their shares, the well-known solution is to let A cut the cake into two pieces and B choose one piece. This forces A to cut the cake into pieces as equal in size as possible since otherwise B would choose the bigger piece. What is sought is a procedure for the generalized problem, under similar circumstances, to divide a cake equally among three or more people subject to the following conditions:

1. The procedure must allow *each* person to have the opportunity of receiving his fair share of the cake regardless of the actions of any other person or group of persons who may have previously schemed to obtain more than their fair share of cake and then to divide it up later.
2. The procedure must involve only a finite number of cuts and steps.
3. Except for temporal ordering, no statement can be conditional or depend on the outcome of any previous statement.
4. No statement can specify in any way the size of a piece or pieces to be cut or chosen.
5. The complete procedure is assumed to be known by all before being carried out.
6. The only allowed operations in the procedure are cuts and choices and combining more than one piece into a single piece. Possible steps, for example, could be (1) A cuts the cake into 6 pieces. (2) B chooses 4 pieces and puts them together and cuts the sum into three pieces. (3) C chooses one piece from A and one from C, etc.

The following solution is from David Evans, who attributes it to Martin Gardner's book, *Aha! Insight*: A knife is moved slowly across the cake. When any of the n participants believes that at least 1/n of the cake has been measured off, that person yells "Cut!", the cake is cut, and that person receives the piece and drops out. This reduces n by one, and the process is repeated until n = 2. Now we have the well-known solution mentioned in the problem. This procedure guarantees that each person believes he or she received a fair piece but does not ensure that no one believes that someone else received a bigger piece. No procedure is known that gives this greater assurance for $n > 3$.

Also solved by Larry Marden, Frank Carbin, Richard Hess, Matthew Fountain, and the proposer, Howard Nicholson.

Better Late Than Never

M/A SD 2 Mary Lindenberg notes that AB should read DB.

Proposers' Solutions to Speed Problems

SD 2 The expansion of 7^4 is 2401. Multiplying 2401 by itself four times, the last three digits become 801, 201, 601, and 001 (adding 400 for each multiplication). Then 7^{20} ends in 001 and also 7 raised to a power which is any multiple of 20, such as 7^{9980}. To get 7^{9984} multiply by 2401 four times and the last three digits become 401, 801, 201, and 601. To get 7^{9999} multiply by 7 three times and the last three digits become 207, 449, and 143—which is the answer.