Puzzle Corner
Allan J. Gottlieb

Painless Number Theory

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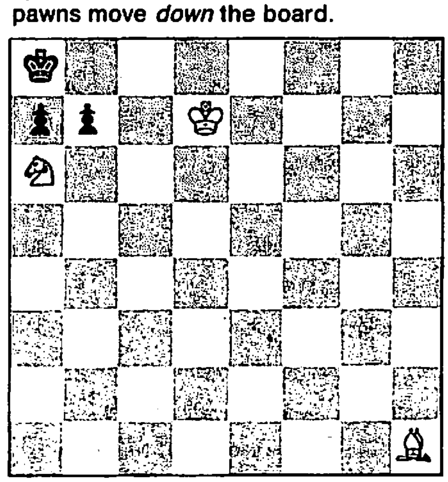
Let me begin with some really good news. At 6:08 p.m. on Sunday, March 14, my wife Alice gave birth to our first child, a nine-pound one-ounce boy. Mother and son are doing fine—father's status is unknown. Since we had participated in a Lamaze training course, I was able to assist during labor and delivery—a truly memorable and fulfilling experience. Although the husband's role in childbirth is relatively minor, Alice says I was very helpful and I believe that I performed my (comparatively easy) functions quite well. Alice of course was the real star and was magnificent. Forgive the new and proud father for rambling on but let me dedicate this column to the fruit of Alice's labor, David Bendix Gottlieb.

Problems

M/J 1 Jerome Taylor sent us the following "double" problem which he attributes to William Crapo:
Given the positions shown, White is to play and mate in three moves. In the first problem, the black pawns move *up* the board, in the second the black pawns move *down* the board.



M/J 2 Roy Sinclair wants to know how many regular decompositions there are of the sphere. A regular decomposition is one in which the surface is subdivided into congruent spherical regular polygons, where a spherical polygon is called regular if all its vertex angles are equal.

M/J 3 Alan LaVergne has a number theory problem—but don't despair, it's not that bad: Given an integer $n>0$ and an odd integer m, prove that any odd integer has one and only one mth root mod 2^n.

M/J 4 The following problem was published in *Technology Review* in 1939 in an advertisement of Calibron Products, Inc., credited to an earlier publication (1922) in the *American Mathematical Monthly* credited to a Professor Shuh of Delft. The problem is to determine the 145 digits in this skeleton division in which each "x" represents one digit. The nine quotient digits shown in bold type form a repeating decimal (i.e., the group as a whole repeats indefinitely). Divisor and dividend have no common factor. Find the digits; there is only one possible answer.

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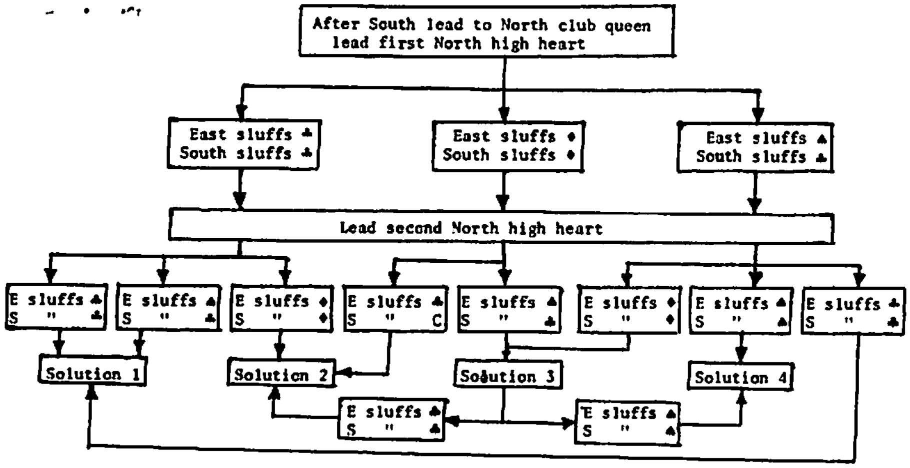
M/J 5 Howard Nicholson has a problem that is really a piece of cake—in fact, several pieces: Consider the problem of dividing a cake equally between two people A and B using only a knife. Assuming A and B always try to maximize their shares, the well-known solution is to let A cut the cake into two pieces and B choose one piece. This forces A to cut the cake into pieces as equal in size as possible since otherwise B would choose the bigger piece. What is sought is a procedure for the generalized problem, under similar circumstances, to divide a cake equally among three or more people subject to the following conditions:
1. The procedure must allow *each*



person to have the opportunity of receiving his fair share of the cake regardless of the actions of any other person or group of persons who may have previously schemed to obtain more than their fair share of cake and then to divide it up later.

2. The procedure must involve only a finite number of cuts and steps.

3. Except for temporal ordering, no statement can be conditional or depend on the outcome of any previous statement.

4. No statement can specify in any way the size of a piece or pieces to be cut or chosen.

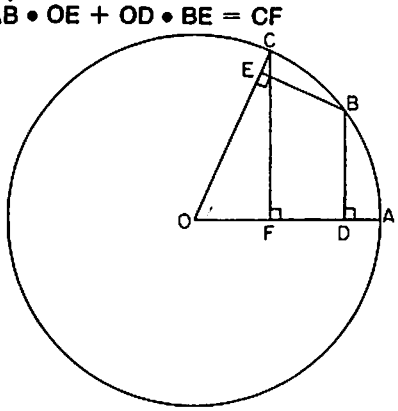
5. The complete procedure is assumed to be known by all before being carried out.

6. The only allowed operations in the procedure are cuts and choices and combining more than one piece into a single piece. Possible steps, for example, could be (1) A cuts the cake into 6 pieces. (2) B chooses 4 pieces and puts them together and cuts the sum into three pieces. (3) C chooses one piece from A and one from C, etc.

Speed Department

M/J SD 1 We are given points A, B, and C arbitrarily located on the quadrant of a circle with BD, BE, and CF perpendicular to OA, OC, and OA, respectively. Under what conditions do the line segments satisfy the following equation?

$AB \bullet OE + OD \bullet BE = CF$



M/J SD 2 Greg Huber wonders what is the maximum amount of change in the form of pennies, nickels, dimes, quarters, and half-dollars you can have and still be unable to change a $1 bill.

Solutions

JAN 1 South, on lead, is to make all seven tricks against any defense, with hearts as trumps:

North: ♠ — ♥ K Q ♦ 5 ♣ Q 10 9 5

West: ♠ 8 7 6 5 ♥ J 5 ♦ — ♣ J

East: ♠ K 10 ♥ — ♦ 10 ♣ 8 7 6 4

South: ♠ J 9 3 ♥ — ♦ J ♣ A K 3

The following graphic solution, including the chart at the top of this column, is from John Boynton:

1. First lead by South has to be a small club to the queen in dummy, since any other lead, except a high club, results in either a trump trick by West or a spade trick by East, unless dummy ruffs, in which case West will ultimately get a trump trick. If South leads a high club first, there is no way to get to dummy without either trumping a spade, which as before yields a spade trick to West, or leading the small club, which is naturally trumped by West.

2. Second lead from dummy has to be a high heart, since a diamond lead is ruffed by West, as is any club lead. Here, declarer must watch East's sloughs carefully. If East makes the mistake of sloughing a club on this first heart lead from dummy, South merely sloughs both high clubs to establish dummy's good clubs, followed by a lead to South's good diamond. If East sloughs a diamond on dummy's first heart lead, South follows with a slough of the ♦J. If East sloughs either spade on the first heart lead from dummy, South still sloughs a high club.

3. If, after a diamond slough by both East and South on the first heart lead from dummy, East sloughs a club on the second heart lead from dummy, South also sloughs a high club and next plays the good ♦5, sloughing South's remaining high club and running dummy's good clubs. If East chooses to slough a spade on the second high heart lead from dummy, South also sloughs the high club and leads the good diamond from dummy. If East now sloughs a club, South follows with a high club slough, establishing dummy's good clubs. If East sloughs the ♠K, declarer is left with a high club in the South hand for an entry to the top two remaining spades.

4. If East sloughs a spade on the first heart lead from dummy, declarer still sloughs a high club from South and watches East's slough on the second heart lead from dummy. If East sloughs a club, the second high club is sloughed from South and dummy's good clubs are taken before leading to the

good diamond in South. If East sloughs spade-diamond on the first two heart leads, declarer leads the good ♦5, after also sloughing South's diamond when East sloughs a diamond, and watches East's critical third slough. If a club is sloughed, South's remaining high club is sloughed, establishing dummy's clubs. If the last spade is sloughed by East, declarer again has an entry to South in the remaining high club to cash the good spades. See where the following four solutions are shown:

Solution 1: Play dummy's good clubs from top, sluffing South's spades; lead to South's good diamond.

Solution 2: Board is good, with high clubs led from top after sloughing South's remaining club on ♦5.

Solution 3: Play dummy's good diamond, watch East's slough: if a club, slough South's remaining high club and proceed with Solution 2; if a spade, slough South's low spade and proceed with Solution 4.

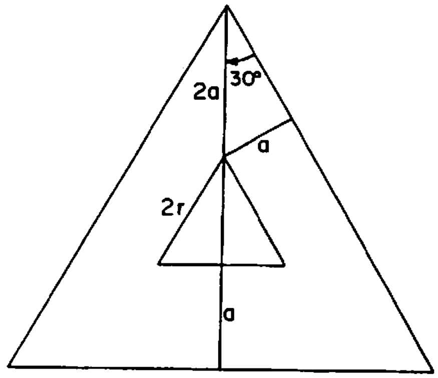
Solution 4: Lead to good South hand via South's high club and play good spades.

Also solved by Charles King, Richard Hess, Matthew Fountain, Edwin McMillan, Charles Rivers, William Katz, Winslow Hartford, Edgar Rose, Carl Peterson, John Rutherford, John Woolston, Howard Katz, Doug Van Patter, Mike Bercher, Manuel Madnick, Avi Ornstein, Matt Bendaniel, Steve Feldman, and the proposer, Emmet Duffy.

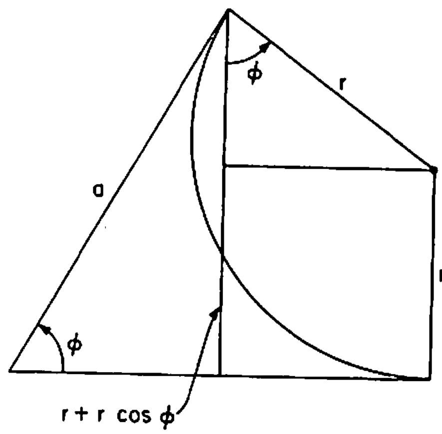
JAN 2 What is the minimum possible volume of a regular tetrahedral box to contain four identical balls?

Selecting a solution for this problem was difficult, since there were several fine responses each with carefully drawn figures. Eventually the dart landed on Charles Rivers' solution:

1. For a regular tetrahedron with sides of length L, Volume = $L^3/6\sqrt{2}$.
Altitude of a side = $\sqrt{3}L/2$.
Altitude of tetrahedron = $\sqrt{2}L/\sqrt{3}$.
Angle of intersection of two adjacent sides (ϕ): $\cos\phi = 1/3$; $\sin\phi = 2\sqrt{2}/3$.
2. Each side is tangential to three spheres of radius r. The points of tangency form an equilateral triangle of side 2r, which is equidistant from the edges of the side by distance a.



Altitude of side = $\sqrt{3}L/2 = \sqrt{3}r + 3a$, or
$L = 2r + 2\sqrt{3}a$.
3. Consider a slice parallel to the altitude and through a point of tangency:



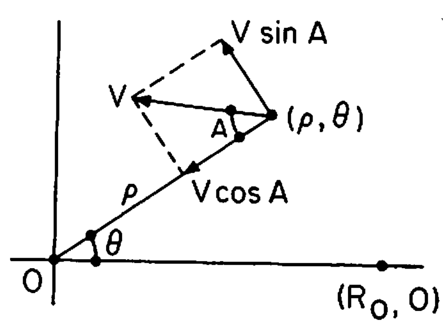
$a = [r(1 + \cos\phi)]/\sin\phi = \sqrt{2}r$.
4. Therefore, $L = 2r + 2\sqrt{6}r = 6.9r$, and
Volume = $(6.9r)^3/6\sqrt{2} = 38.7\ r^3$.

Also solved by John Woolston, Harry Zaremba, Norman Wickstrand, Avi Ornstein, Emmet Duffy, Lyndon Welch, Richard Hess, Irving Hopkins, Matthew Fountain, and the proposor, Rob Cave.

JAN 3 *Technology Review* apologizes for the absence of a solution to JAN 3 in this issue. Due to circumstances beyond the author's control, that solution is being held for publication in "Puzzle Corner" for July, 1982.—*Ed.*

JAN 4 A rocket is launched with constant speed from the point (R_0,O) toward a stationary target located at (O,O). The rocket has a seeker that is offset from the velocity vector of the rocket by a constant angle A. The seeker always points directly at the target. What is the flight path of the rocket and what is its acceleration?

The following solution is from Harry Zaremba:



In the figure shown, v is the rocket velocity and (ρ, θ) are the polar coordinates defining the position at any instant of time t.
The radial velocity toward the target will be
$d\rho/dt = \dot{\rho} = -v\cos A$ (1)
and the velocity perpendicular to ρ will be
$\rho\,(d\theta/dt) = \rho\dot{\theta} = v\sin A$ (2)
Dividing (1) by (2),
$d\rho/\rho d\theta = -(\cos A)/(\sin A) = -1/(\tan A)$, or
$(d\rho)/\rho = -d\theta/(\tan A)$. (3)
Integrating (3),
$\log\rho = -\theta/(\tan A) + \log k$, or
$\rho/k = e^{-\theta/(\tan A)}$ (4)
Noting that $\theta = 0$ when $\rho = R_0$, then from (4), $k = R_0$.
Hence, the flight path of the rocket is
$\rho = R_0 e^{-\theta/(\tan A)}$.
Differentiating (2) with respect to t,
$\rho\ddot{\theta} + \dot{\theta}\dot{\rho} = 0$. (5)
Substituting $\dot{\rho}$ and $\dot{\theta}$ from (1) and (2) into (5),
$\rho\ddot{\theta} + (v/\rho\cdot\sin A(-v\cos A) = 0$.
Thus, the angular acceleration of the rocket with respect to the target position is
$\ddot{\theta} = (v/\rho)^2\sin A\cos A$.
Since the radial velocity is constant, the rocket's time to reach the target is
$t = R_0/(v\cos A)$.

Also solved by Matthew Fountain, Richard Hess, Emmet Duffy, Randall Gressang, John Woolston, Michael Jung, and the proposer, Tom Hafer.

Better Late Than Never

JAN SD 1 Richard Desper disagrees with the published solution; Elliot Roberts suggests a windmill-driven propeller; and Richard Russell writes:
The published solution seemingly violates the basic laws of vector analysis. Consider a sail as a flat plane, set, for example, 30° offwind. This locally changes the direction of the wind and sets up a resultant force on the plane that is at right angles to that plane or 120° offwind. However, if the plane is free to move 45° relative to the wind source, then the 120° resultant force can be represented by two force components or vectors, one at 45° to the wind and one at 135° to the wind. Graphically we can see that the 45° vector drives the boat forward in the direction of the 45° vector and that the 135° vector neither helps nor hinders such movement. Now without changing the angle of the attack of the plane (30°), we can move the plane on a course that is closer to the wind. Inspection shows that the forward-drive component becomes less, reaching zero by the time the movement coincides with the plane at 30°. If the movement is brought still closer to the wind, then the resultant component is negative or drives the system back away from the wind as would be expected. Generalizing: 1. The sail plane has to be angled to the wind. 2. The movement of that sail plane has to be at a greater angle than the sail plane itself. 3. Given conditions 1. and 2., the boat will move forward. 4. As both conditions 1. and 2. approach 0, condition 3. remains valid if friction is 0.

JAN SD 2 Al Weiss, Everett Leroy, and Chester Tudbury found other solutions.

A/S1 Fred Trescott notes that the position cannot occur in a game.

A/S4 Irving Hopkins notes that no one thought of the circumscribing circle.

OCT 3 L. Steffens found a simpler solution and Ahmet Karakas has responded.

OCT 4 Greg Huber believes that the published solution neglects a term involving the central tetrahedron's edges. He obtains V = 0.5402619.

OCT 5 Ahmet Karakas has responded.

N/D 2 Ed Chang and Mary Lindenberg have responded.

N/D SD 1 John Kellam found 100 triangles.

FEB 4 John Rule disavows authorship.

Proposers' Solutions to Speed Problems

SD 1 When the radius is 1.
SD 2 $1.19.