Bags of Coins: Are They True or False?

Allan Gottlieb studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973); he has just been promoted to associate professor in the Mathematics Department (where he is also coordinator for computer mathematics) at York College of the City University of New York. Send problems, solutions, and comments to him at the Department of Mathematics, York College, Jamaica, N.Y., 11451.

This is written on January 1, so let me wish all of you a (belated) happy new year. 1978 was a good year for me, and I am looking forward to 1979; and I hope you feel the same way.

In December I attended the Association for Computing Machinery annual conference in Washington, D.C., and as usual the North American Computer Chess Championship was held concurrently. To the surprise of many (your editor included), the current world champion, CHESS 4.7 from Northwestern University, was soundly beaten by BELLE, from Bell Labs. This latter program, written by Ken Thompson, runs on a minicomputer, the PDP-11 model 70, whereas CHESS 4.7 utilizes a supercomputer, the CDC CYBER 176: This was not, however, a David vs. Goliath encounter. The BELLE computer was greatly enhanced by some chess-specific hardware built by Joe Condon, also from Bell Labs. The level of play produced by the top contestants was quite high, at least class A. BELLE, itself, must surely be in the expert range. Although some optimists are starting to talk about a computer as world champion within ten years, I stick with my prediction that 20 years would be closer.

Problems

M/A 1 A cute cryptarithmetic problem from the late R. Robinson Rowe:
Evaluate this 4-story gabled HOUSE on its GROUNDS so generous that there is both a front yard and a back yard.

```
    U
   O U S
 H O U S E
 H O U S E
 H O U S E
 H O U S E
-----------
G R O U N D S
```

M/A 2 Here's a bridge problem from Noland Poffenberger, who wants South, on lead, to make the remaining six tricks with hearts as trump:

North: ♠— ♥8 7 ♦J 5 ♣A 2

West: ♠Q 7 ♥— ♦Q 7 ♣J 3

East: ♠J 6 ♥— ♦10 8 6 ♣8

South: ♠10 9 ♥— ♦A K 9 ♣10

M/A 3 Emmet Duffy sent us the following counterfeit problem:
Eleven open bags, numbered 1 to 11, each contain two coins. Ten bags contain genuine coins all of the same weight. One bag contains false coins which differ slightly in weight from genuine coins. The two false coins weigh the same but a false coin is either 10 grains heavier or 10 grains lighter than a genuine coin. You are given a balance scale and 20-grain weight. Using the balance scale two times only, find a method to identify the false coins and tell whether they are heavy or light. Placing stated coins on stated sides of the scale, and, if required, placing the weight on the scale, shall be considered a use of the scale.

M/A 4 Avi Ornstein and Hugh Thompson each submitted the same variant on 1975 DEC 3:
Find a collection of (ordinary) English words that contains the fewest possible total number of letters while including each of the 26 letters at least once.

M/A 5 Baranow needs some help in unicycling across a rotating table:
A unicyclist moving at a velocity v enters the north end of a table rotating at an angular velocity w. He wishes to leave via an exit ramp at the south end of the table in the minimum possible time, t. Find t and the path required as a function of v, w, and r (the radius of the table).

Speed Department

M/A SD 1 Here is another problem that originally appeared in *Technology Review* in advertising for Calibron Products — this one from February, 1942 (*diagram is to the right*):
A reel capable of holding 4,000 feet of steel tape 0.003 inch thick is to be designed. If the inner (hub) radius is 4 inches, can you derive (in your head) a formula and rough numerical value for the outside radius R?

M/A SD 2 Joe Horton wants to know what algebraic expression (involving a limit) defines a square.

Solutions

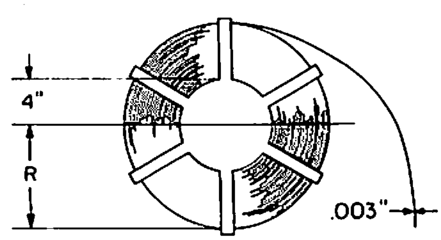
J/J 1 (as modified in November)

North: ♠7 3 ♥Q J 3 2 ♦5 2 ♣A K 5 3 2

South: ♠A Q 2 ♥A K 7 6 5 4 ♦A Q ♣J 6

With South the declarer at six hearts, West leads the ♥10. South thinks a moment or two and plays a heart from dummy. East discards a diamond. What is the best play to make six hearts? If possible, supply the probability of success.

The proposer has a very detailed solution with a 95.7-per-cent probability of success. Due to space considerations, however, I print below Matthew Chen's solution instead, even though the latter has a probability of success nearer to 95.1 per cent:
Trick 1: ♥J, ♥4. Trick 2: ♠3, ♠Q (if ♠K doesn't appear); if the finesse succeeds, South draws trumps and claims. Otherwise, say West returns a heart; if spades are 8-0, South draws trumps and takes the diamond finesse (after testing if ♣Q drops singleton). Else, South plays trick 3: ♥2, ♥K. Trick 4: ♣6, ♣K. Trick 5, assuming ♣Q did not drop: ♣A, ♣J. Trick 6: ♣2, ♥A. Trick 7: ♥5, ♥Q. Trick 8: ♣3, ♥6. If West can ruff at trick 4 or 5, then the contract fails. If clubs split 2-4, 3-3, or 4-2, then South cashes ♠A, ruffs a spade, and discards ♦Q on the last club. If clubs are 5-1 or 6-0, South takes the diamond finesse (unless ♣Q is a singleton). At first, the possibility of a singleton ♣Q is ignored in the following analysis of South's winning chances. South wins whenever the ♠K is favorably located, unless all of the following occur: spades break 0-8,



clubs split 5-1 or 6-0, and West holds ♦K. West holds 10 unknown cards, and East has 12 unknown cards, so the probability that (East holds the ♠K and South can win) is

$$\frac{\binom{21}{11}}{\binom{22}{12}} - \left(\frac{\binom{8}{0}\binom{6}{6}\binom{7}{3}}{\binom{22}{10}} + \frac{\binom{8}{0}\binom{6}{5}\binom{7}{4}}{\binom{22}{10}} \right)$$

$$= \frac{6}{11} - \frac{35}{92378} \approx .545076$$

If the ♠K is offside, South wins when clubs are 2-4, 3-3, or 4-2, and also when (clubs are 5-1 or 6-0 *and* ♦K are on side). After trick 2 South knows whether spades are 8-0; if they are not, West has nine unknown cards and East 11 unknown cards. So let E_1 be the event that West holds ♠K, E_2 that spades are 8-0, E_3 that East holds ♦K, E_4 that clubs are 2-4, 3-3, or 4-2, E_5 that clubs are 5-1 or 6-0, and $\bar{E}_2$ that E_2 does not occur. Then the probability that (E_1 obtains and South can win) is

$$P(E_1)\{P(E_2|E_1)P(E_3|E_2E_1) + P(\bar{E}_2|E_1)$$
$$[P(E_4|\bar{E}_2E_1) + P(E_5E_3|\bar{E}_2E_1)]\}$$

≈ 0.395509, since

$P(E_1) = 5/11$,

$$P(E_2|E_1) = \frac{\binom{7}{7}\binom{14}{2}}{\binom{21}{9}} = \frac{1}{3230},$$

$$P(E_3|E_1E_2) = \frac{\binom{13}{\alpha}}{\binom{14}{2}} = \frac{6}{7},$$

$$P(\bar{E}_2|E_1) = \frac{3229}{3230},$$

$$P(E_4|\bar{E}_2E_1) = \frac{\binom{6}{2}\binom{14}{7}}{\binom{20}{9}} + \frac{\binom{6}{3}\binom{14}{6}}{\binom{20}{9}} + \frac{\binom{6}{4}\binom{14}{5}}{\binom{20}{9}},$$

$$P(E_5E_3|\bar{E}_2E_1) = \frac{\binom{6}{5}\binom{13}{4}}{\binom{20}{9}} + \frac{\binom{6}{6}\binom{13}{3}}{\binom{20}{9}}.$$

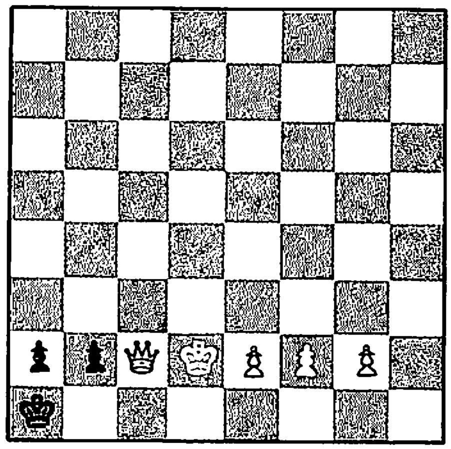
Hence South makes his contract with probability $\approx$ .9406 plus P ((spaces are 0-8 and clubs are 5-1 with ♣Q singleton, and West holds ♦K) or (West holds ♠K and ((clubs are 5-1 with ♣Q singleton and West holds ♦K) or (clubs are 1-5 with ♣Q singleton)))).

$$.9406 + \frac{\binom{7}{4}}{\binom{22}{10}} + \frac{\binom{14}{3}}{\binom{22}{10}} + \frac{\binom{15}{8}}{\binom{22}{10}}$$

$\approx$ .9406 + .01056838 $\approx$ .95115 (exact to five digits).

Also solved by Matthew Fountain, Matthew Chen, Elmer Ingraham, Rudolph Evans, Andrew Purbrick, and the proposer, William Butler.

NOV 1 White is to move and win:



Despite the fact that we mistakenly stated that "White moves down the page," many readers solved this problem. They all agree that the solution is:

1	Q — B3	K — N8
2	Q — Q3*	K — R8
3	Q — Q4	K — N8
4	Q — K4*	K — R8
5	Q — K5	K — N8
6	Q — B5*	K — R8
7	Q — B6	K — N8
8	Q — N6*	K — R8
9	Q — N7	K — N8
10	Q — R7*	K — R8
11	Q — R8	K — N8
12	Q — R1 mate	

Solutions received from Smith Turner, Gerald Blum, Richard Denton, Jordan Wouk, Jon Thaler, Bill Camperlino, Abraham Schwartz, William Butler, Winthrop Leeds, Matthew Fountain, Robert Kimble, and the proposer, Steve Slesinger.

NOV 2 Find all positive integer solutions of $A^2 + (A + 1)^2 = C^2$.

Although several readers gave short "proofs" that 3, 4, or 5 is the only solution, there are, in fact, infinitely many others. Steve Feldman noted that a detailed discussion can be found in Beiler's *Recreation in the Theory of Numbers* (Dover, 1974), and Emmet Duffy submitted the following:
Noting that the proposer wants to know all positive integer solutions but not how to get them, we do not have to use the Pell equation. There are an infinite number of solutions; the first two values of A are 3 and 20 and for C, 5 and 29. The values of A are a series where any number (after the first two) is found by multiplying the previous number by 6, subtracting the number before the previous number, and then adding 2. The value of C is found in a similar manner except do not add 2. The first ten values of A and C are:

A	C
3	5
20	29
119	169
696	985
4,059	5,741
23,660	33,641
137,903	195,025
803,760	1,136,689
4,684,659	6,625,109
27,304,196	38,613,965

Also solved by Matthew Fountain, Frank Carbin, Winthrop Leeds, William Butler, Wayne Adams, Grant Sharman, Roger Milkman, Harry Zaremba, Winslow Hartford, John Prussing, Judith Longyear, Gerald Blum, Smith Turner, Jack Crawford, Randall Rathbun, Robert Kimble, Neil Cohen, Avi Ornstein, Gerald Stills, William Pasfield, Robert Kennedy, and the proposer, Thomas Mahon.

NOV 3 In tennis, to win a game you must meet two conditions: a) win four or more points, and b) win two more points than your opponent — thus (dispensing with the fancy terminology of tennis scoring) winning scores are 4-0, 4-1, 4-2, 5-3, etc. If one player's probability of winning any given point is p, what is his probability of winning any given game? If winning a set requires winning at least six games, and at least two more games than your opponent, what is this player's probability of winning any given set?

The following solution is from Gerald Blum:
If one/player's probability of winning a point is p, that player's probability of winning a game is given by

$$g = p^4 + 4p^4(1 - p) + 10p^4(1 - p)^2$$
$$[1 - 2p(1 - p)]^{-1},$$

where the term in brackets is the sum of an infinite series. The first term is the probability of winning 4-0, the second the probability of winning 4-1 (the winning player must win the last point), and the coefficient of the bracket is the probability of winning 4-2. The probability of winning 5-3 is the probability of winning the last two points after getting to a 3-3 deuce; it happens to be $2p(1 - p)$ times the probability of winning 4-2. The probability of winning $(n + 2)$ to n is the probability of winning the last two points after being at deuce at every point from 3 to n. To get from one deuce to the next, each player must win one point, but in either sequence; the probability of this is $2p(1 - p)$. The probability of winning $(n + 2)$ to n is thus the quantity $2p(1 - p)$ all to the $(n - 3)$ power times that for winning 5-3, and thus the quantity $2p(1 - p)$ all to the $(n - 2)$ power times that for winning 4-2. Summing over all possible winning scores gives an infinite geometric series with the sum indicated above. We can combine terms to give

$$g = (15 - 34p + 28p^2 - 8p^3)p^4/$$
$$(2p^2 - 2p + 1).$$

With some algebra, one can confirm that replacing p by $(1 - p)$ and g by $(1 - g)$ everywhere above gives the same answer. Following essentially the same logic, one gets for the probability of winning a set

$$s = g^6 + 6g^6(1 - g) + 21g^6(1 - g)^2$$
$$+ 56g^6(1 - g)^3 + 126g^6(1 - g)^4$$
$$[1 - 2g(1 - g)]^{-1}.$$

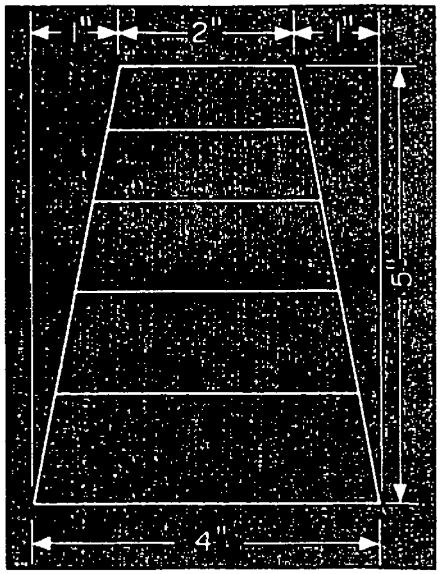
One could, of course, express s in terms of p by substituting, but there's a limit to how much slog even I will go through!

Also solved by Randall Rathbun, Jack Crawford, Smith Turner, Judith Longyear, Winslow Hartford, Harry Zaremba, Sidney Shapiro, Ernesto Ramos, William Butler, P. Jung, Matthew Fountain, Michael Fuerst, Wayne Baxter, and the proposer, Edwin Strauss.

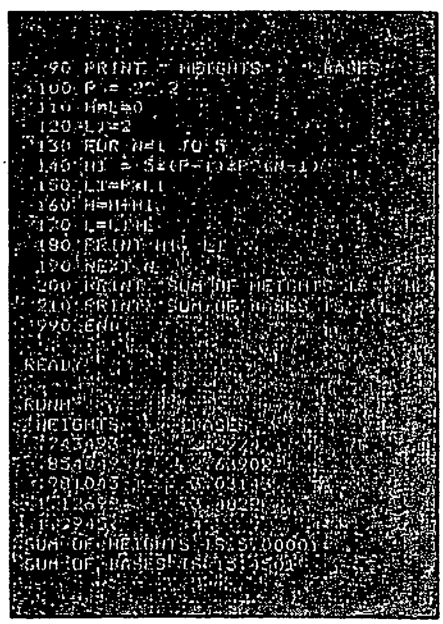
NOV 4 The figure shows five similar trapezoids formed by drawing lines parallel to the base. What are the heights of the trapezoids? *(see diagram on top of the next column)*

James Boettler sent us the following solution, which he says was done "while watching 'Heroes of the Bible' on television"; he includes a BASIC program to do the calculations:

Draw a vertical line from the top of the trapezoids to the bottom, and observe that



the tangent of the angle between the sides of the trapezoids and the vertical line is 1/5. Equate this to another expression for the tangent of the angle. Since the trapezoids are similar, the ratio of the sides must be the same. Combining these observations leads to the formula $h_n = 5(P - 1)P^{n-1}$, where P is the fifth root of 2, h_n the height of the nth trapezoid. Also, $L_n = PL_{n-1}$ for the base of successive trapezoids. Here is the BASIC program:



Also solved by Mike Fuerst, Homer Stewart, Matthew Fountain, Richard Mackler, Jerome Taylor, Bill Marshall, Winthrop Leeds, Norman Wickstrand, Everett Leroy, William Butler, Arthur Hovey, Frank Carbin, Charles Norton, Andrew Combie, Sidney Shapirö, Temple Patton, Harry Zaremba, Mary Lindenberg, Jon Thaler, Winslow Hartford, John Prussing, Judith Longyear, T. Benton, Gerald Blum, Smith Turner, Elwyn Adams, Art Porter, Jack Crawford, Peter McMenamin, Avi Ornstein, Gerald Stills, Naomi Markovitz, Randall Rathbun, Roger Milkman, and the proposer, Emmet J. Duffy.

NOV 5 What is the English word with the largest number of consecutive letters appearing in alphabetical order?

Looking at everyone's results, I find that the largest number of consecutive letters is four. Examples include "understudy," "overstuffed," "limnophilous," and "gymnopedia." Solutions were received from Winthrop Leeds, Steve Feldman, P. Michael Jung, Daniel Bloom, Cary Silverston, D. Terence Langendoen, Gerald Blum, Emmet Duffy, Randall Rathbun, Arthur Delagrange, and the proposer, Dave Rabinowitz.

Better Late Than Never

1977 DEC 5 Edwin Comfort has responded.

1978 M/A 2 E. Phillips found the same improved solution noted in this department for December/January, and Albert Mullin submitted the following:

The "factor champions" discussed by Engel were discovered by the great Indian arithmetician S. Ramanujan about the year 1913. He called them "highly composite numbers." Using sophisticated techniques from real analysis, he studied their properties in a lengthy paper published in the *Proceedings of the London Mathematical Society* in 1915. On the other hand, Engel may have the *longest* list of highly composite numbers in existence. To this extent, it is just possible that the Indian Mathematical Society might be interested in the "complete" list. However, in recent years that society has expressed more interest in abstract matters and just might *not* be interested in such a list of numbers.

M/A 5 Winslow Hartford has responded.
J/J 4 Avi Ornstein has responded.
J/J 5 T. Benton and Elwyn Adams have found ruler-and-compass constructions.
A/S 1 Mike Bercher and Von Fischer have responded.
A/S 3 Leon Bankoff has a solution that does not require trigonometry.
A/S 4 W. Stiehl, Walter Nissen, and Jack Crawford have found several specific solutions. Mr. Nissen is the leader with the

following complete list up to one billion: 1, 2, 3, 4, 5, 6, 7, 8, 9, 153, 370, 371, 407, 1634, 8208, 9474, 54748, 92727, 93084, 548834, 1741725, 4210818, 9800817, 9926315, 24678050, 24678051, 88593477, 146511208, 472335975, 534494836, 912985153.

OCT 1 Timothy Maloney and Jack Crawford have responded.
OCT 2 Elwyn Adams and Mahlon Stilwell have responded.
OCT 3 Mahlon Stilwell has responded.
OCT 4 Mahlon Stilwell, Joe Stockert, Von Fisher, Judd Schwartz, and Jon Thaler have responded. Mary Lindenberg noticed that this problem was also 1972 O/N 2.

Proposers' Solutions to Speed Problems
M/A SD 1 Equate the area of the thin edge of the tape (114 square inches) to π * (R*R − 4) and solve for R. The answer is approximately eight inches.
M/A SD 2 $\lim_{n \to \infty} x^{2n} + y^{2n} = 1$.

Cowen

Continued from p. 11

Admittedly, the flashing lights seen at an angle can look wierd. Some perceptual mistakes probably are inevitable. However, Mr. Hendry says: "None of this rationale begins to 'explain' the distorted observations regarding 'domed discs,' 'treetop heights,' gigantic size [large as a football field] estimates, claims of being deliberately followed in cars, false assumptions that the plane's sign turning off was equal to the U.F.O. rushing away faster than the eye could follow . . . and, most of all, the wholly unwarranted emotional reactions (including fear) exhibited by the eyewitnesses and the immediate, nearly universal conclusion that the advertising plane was from outer space." Mr. Hendry concluded that this is indicative of "pervasive emotional climate that appears to be surrounding the entire U.F.O. subject, one that succeeds in distorting even the most commonplace sightings into exaggerated 'miracles.' "

It is this emotionalism that seems so puzzling. Dr. Leibowitz says he has no scientifically based explanation for it. However, he thinks it may be at least partly due to widespread distrust of science and consequent desire to put science down. "I think society in general thinks science has gotten too big and many people like to think science doesn't have all the answers," he says.

He may be right. Certainly, the irrational belief in flying saucers involves rejecting scientific method and standards of evidence and credibility. There is a research challenge here for social scientists to find out just what is involved. As Mr. Hendry points out, the I.F.O.s (Identified Flying Objects) offer "a wealth of evidence . . . about U.F.O.logy's old bugaboo: the reliability of human testimony." Meanwhile, it's worth recalling the biblical advice (Proverbs 14:15): "A simple man believes every word he hears; a clever man understands the need for proof." □