How's Your Calculus?

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As a new volume begins, let me review the ground rules of "Puzzle Corner" for new readers.

Each issue we publish five regular problems and two "speed" problems. Three issues later the solutions to the regular problems appear. This month, for example, we are printing the solutions to problems published last May. Challenges to published solutions and acknowledgement of late responses appear in the "Better Late Than Never" department. The "speed" problems are not to be taken too seriously. Often whimsical, their solutions are usually given the same issue as the problem is posed, and they rarely appear in the "Better Late Than Never" department.

Here is some news from our readers:

I remember that during my senior year at M.I.T., many of the graduating seniors were considering their chances for acceptance at various graduate schools. One of my friends, Mike Rolle, decided to enhance his chances by solving the famous four-color conjecture. Since generations of mathematicians had failed in this attempt, we didn't feel that Mike had much hope of success; but he was serious. During that year he actually obtained some impressive partial results, but the conjecture was still unsettled. The end of this story occurred this year after Appel and Haken finally solved the problem. I was reading their important papers in the *Illinois Journal of Mathematics* and noticed an acknowledgement to one Michael Rolle for his help. Congratulations, Mike; he who laughs last . . .

Congratulations, too, to Frank Rubin: the *Journal of Recreational Mathematics* will have a special Frank Rubin issue next April. . . . Dale Overy (27 Bodmin Avenue, Stafford Staffs ST17 OEF, England) has started a newsletter called *Puzzle World* devoted to mechanical puzzles.

Judith Longyear has suggested an informal poll of our readers to see which kinds of problems (chess, bridge, geometry, cryptarithmetic, etc.) are most (and least) appreciated. You are all welcome to respond, and significant preferences will effect future problem selections.

Several readers noted an error in the published solution to NS9. That problem is now reopened.

Problems

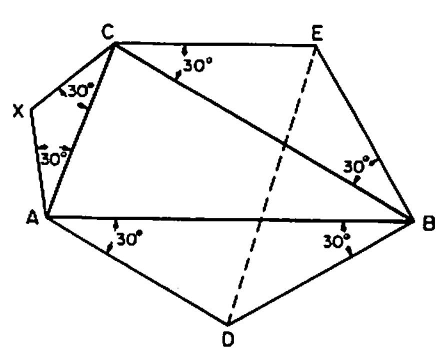
OCT 1 William Butler wonders how normal is normal (in bridge, at least): Conventional point counting gives four points for each ace, three per king, two per queen, and one per jack. The average bridge hand has ten such points. What is the probability of receiving a hand with exactly ten points?

OCT 2 Sebastian Batac would like to find two positive rational numbers the sum of whose cubes is 6. In other words, find positive integers a, b, c, and d satisfying $(a/b)^3 + (c/d)^3 = 6$.

OCT 3 The following cryptarithmetic problem from Avi Ornstein consists of two mathematical statements which are correct in base 10 when digits are substituted for letters and is also true as read for modulo 9 mathematics:

SIX + TWO + TWO = ONE
SIX + SIX = TWO + ONE

OCT 4 J. Friedman sends me a number of problems published by Calibrom Products as advertising in *Technology Review;* this one appeared in 1938:



Starting with *any* triangle, construct three exterior triangles having base angles of 30° and vertices at D, E, and X — as indicated in the diagram. If the distance DE is taken as 100, what is the distance DX? (The answer is a definite number, not a formula.)

OCT 5 How's your calculus? Harvey Elentuck asks for the area of the loop of $Y^2 = (X + 4)(X^2 - X + 2Y - 4)$.
A noncalculus solution to this would be very impressive, but calculus is permitted.

Speed Department

OCT SD 1 Ruth Duffy asks us to name a word in the English language with seven letters, five of which are the vowels a, e, i, o, and u (but not necessarily in alphabetical order).

OCT SD 2 The Editor of the *Review* discovered the following problem being distributed as part of a tongue-in-cheek "quiz" prepared by M.I.T. students for exhibitors in the 1978 Massachusetts Science Fair:
Translate into a limerick:

$(12 + 144 + 20 + 3\sqrt{4})/7 + 5 \times 11 = 9^2 + 0.$

Solutions

NS 10 (This was first published as 1974 M/A 2 and never solved; it was published again as NS 10 in February, 1978:
Find a closed form for

$1^1 + 2^2 + \ldots + n^n.$

When this was first published Leo Epstein supplied some asymptotic formulas. He has improved these, but we still have no exact closed form. Perhaps none exists.

NS 12 A standard deck of 52 cards is shuffled and placed face down upon the table. The cards are then turned face up one at a time by flipping over the top card of the face-down stack. As this is done, the player simultaneously calls out the sequence A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A, 2, etc., one call being made for each card flipped over. To win the game, one must go through the deck without matching a card flipped over with the card called. Suits don't matter, so, for example, any 4-spot flipped over on the 4th, 17th, 30th, or 43rd turn results in a loss. "Since winter will surely come again," Mr. Connine would like to know what are the chances of winning the game. How about a second solution for the same game with a 48-card pinochle deck?

This problem is not trivial! Judith Longyear gave a colloquium talk on her results in 1974. The answer is not $(12/13)^{52}$. Although the probability of success for any one card is 12/13, the events are

not independent. Bob Kimble and an unnamed computer assert that of the 52! possible decks exactly 1, 309, 302, 175, 551, 177, 162, 931, 045, 000, 259, 922, 525, 308, 763, 433, 362, 019, 257, 020, 678, 406, 144 are winners.
They also solved the pinochle problem: of the 48! decks 2, 173, 013, 719, 746, 911, 580, 113, 686, 677, 997, 894, 282, 336, 936, 761, 753, 600, 000, 000 are winners. Since 52! = 80, 658, 175, 170, 943, 878, 571, 660, 636, 856, 403, 766, 975, 289, 505, 440, 883, 277, 824, 000, 000, 000, 000, they obtain a success probability of about 1.62 per cent.

Stephen Flaum and a TI 58 used an iterative technique:
At each iteration the probability of losing on that iteration is calculated. In addition, the expected number of cards remaining with each face value after the iteration, conditional on the assumption that the game is not lost on that iteration, is calculated. These expected values are used in subsequent calculations of the probability of failure.

This method requires fractions to be kept throughout. Flaum and TI actually divide out the fraction and use the approximating decimal. Perhaps this explains their answer of 1.77 and .0225 percent for pinochle.

A. Walther claims the answer is:

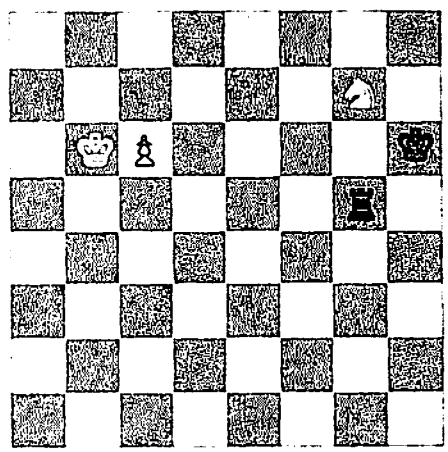
$$\left(\sum_{P=0}^{13}(-1)^{P}\cdot 1/P!\right)^4$$

or approximately e^{-4} (i.e., over 1.8 per cent). His remarks follow:
Make on the table an array 13 blocks long and four blocks wide. Label the four rows with the names of the four suits. Turn the cards over, one at a time, and place them in the array, going from left to right and placing each card in the row matching its suit. After we have gone through the entire deck, we have on the table four rows of 13 cards, one for each suit in the deck. To win the game no card must be in its proper place — i.e., each row of 13 cards must be a "complete permutation." A complete permutation is defined as a permutation in which none of the elements is in its proper place. The theory of complete permutations is developed on some sheets saved for me by E. L. O'Neill which you may want to share with interested readers. The ratio of the number of complete permutations to the total number of permutations for m elements is

$$\sum_{p=0}^{m}(-1)^{p}\, 1/P!$$

This is very nearly e^{-1}; therefore, the answer to the problem is e^{-4} — i.e., about one in 53. (A copy of the notes on complete permutations may be obtained from the editor on request.)

MAY 1 White to play and win:



Several readers slipped up on this one. By playing 2K — B5 they allow a neat draw:

2.	. . .	R — N8
3.	3P — B8 (Q)	R — B8 ch

Joseph Seo, however, avoids this and finds a solution with only one line:

1.	P — B7	R — N3 ch
2.	K — N5	R — N4 ch
3.	K — N4	R — N5 ch
4.	K — N3	R — N6 ch
5.	K — B2	R — N7 ch
6.	K — Q3	R — N6 ch
7.	K — Q4	R — N5 ch
8.	K — Q5	R — N4 ch
9.	K — Q6	R — N3 ch
10.	N — K6	R — N1
11.	N — Q8	R — N3 ch
12.	K — K7	R — N2 ch
13.	N — B7 ch	K — any
14.	P — B8 (Q)	Resigns

Responses also received from Bob Kimble, William Butler, Rufus Franklin, Cary Silverston, Robert Bart, Smith Turner, Jerome Taylor, Winthrop Leeds, Roger and Paul Milkman, Jacob Bermann, T. Mahon, Darryl Hartman, Richard Kandziolka, Richard Hess, Walther Fischer, and Harry Nelson.

MAY 2 "Dentification" and "identification" are both English words. For each English letter α, what is the longest string β such that both β and $\alpha\beta$ (the concatenation of α followed by β) are English words? Pairs such as "allelujah" and "hallelujah" or "enanthaldehyde" and "oenanthaldehyde" are excluded, since they are simply variant spellings or variant pronunciations of the same word.

Dennis Kluk submitted a list which will be hard to beat. (I must add that some of his words are not in my vocabulary; perhaps I should have specified a (small) dictionary in which all words were required to appear.) Mr. Kluk's list, which follows, comes from *Word Ways*, subtitled the journal of recreational linguistics: **Aquintocubitalism, Blithesomeness, Chemotherapeutics, Demulsification(s), Emotionlessness, Frightfulness, Gastrophotographics, Hedriophthalmous, Identification(s), Japaconitine, Kineasthetic, Limitableness, Methylhydrocupreine, Neopaleozoic, Oesophagostenosis, Premosrepresentation(s), Quinta(s), Revolutionally, Selectiveness, Treasonableness, Utopographer(s), Vindictiveness, Whenceforward, Xanthosiderite, Yourselves, and Zoosporiferous.**

Also solved by Harry Hazard, Jacob Bermann, Emmet Duffy, Paul Hertz, and the proposer, Donald Forman.

MAY 3 Given an n-by-n checkerboard and n^2 checkers of n different colors, and given that there are n checkers of each color, is it possible to arrange all the n^2 checkers on the board such that no two checkers of the same color lie in the same row, column, or diagonal? (By "diagonal" is meant *all* the diagonals, not just the two main diagonals.) It turns out that for certain values of n it is possible to so arrange the checkers; in this case we say a solution exists — e.g., n = 1. But for certain other values of n such an arrangement is impossible — i.e., no solution exists. For which values of n does a solution exist?

For some unknown reason, I published this problem twice: once as FEB 3 and now again as MAY 3. More surprising than this is the fact that no reader noticed my error; as soon as I saw the May issue I made ready for the slings and arrows. The responses to MAY 3 are consistent with the published solution to FEB 3 (see *June/July, page 27*). In short, an algorithm exists for $N = 6K \pm 1$ (i.e., N not divisible by 2 or 3), and several readers assert (without proof) that no solution exists for the remaining cases. I repeat my comment of June/July: this looks like an NS problem for the 1980s.

Responses received from Judith Longyear, William Butler, T. Mahon, Ari Ornstein, Richard Hess, Bob Leisy, and the proposer, Sheldon Razin.

MAY 4 Assign numerical values to each letter:

```
          HEN
AARON)PHARAOH
      AARON
       OOYPO
       BBNYZ
       AEYDNH
       ACEPHH
        ERPZC
```

This is a base-12 cryptarithmetic problem, and those solving it were reminded that duodecimal notation has two extra digits following 9 before reading the radix. For uniformity, these were specified to be "dek" (symbol X, numerical value equals decimal 10) and "el" (symbol ϵ, numerical value equals decimal 11). Then the radix is "dozen," or "do" for short.

Cryptarithmetic problems tend to be popular, and this one, with its base-12 twist, was no exception. Several readers asked for more such problems; the best way to achieve this is to send more in. The following solution is from Shirley Wilson:

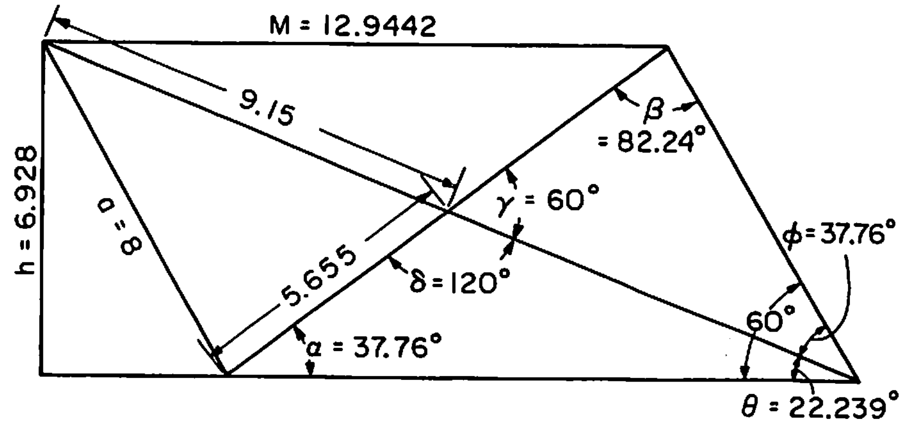
```
          12ϵ
3359ϵ)4135391
      3359ϵ
       99749
       66ϵ7x
       3278ϵ1
       302411
        254x0
```

1. H = 1
2. C = 0
3. A − N = P or 10 (do) + A − N = P, but P = A + 1, so 10 + A − N = P and hence N = ϵ
4. Z = x
5. O = 9
6. Since A − R = 9 or 10 + A − R = 9, and since A ≠ x and A ≠ ϵ, 10 + A − R = 9 and hence 10 + H − 1 − A = 9. Therefore, A = 3 and R = 5.
7. P = 4, E = A − 1 = 2, D = 8, Y = 7, and B = 6.

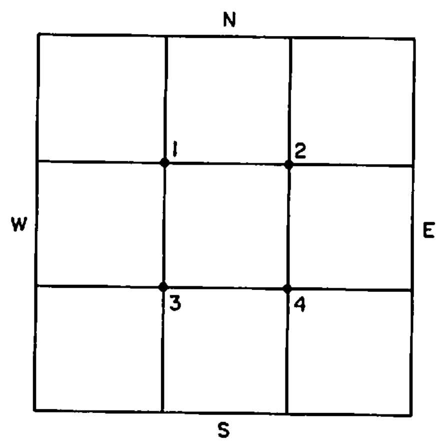
Thus, as many readers noticed, the substitutions are:

0	1	2	3	4	5	6	7	8	9	X	ϵ
C	H	E	A	P	R	B	Y	D	O	Z	N

Also solved by: Rona Rybstein, Bob Kimble, William Butler, Robert Slater, Paul Hertz, Christopher Roth, Robert Bart, George Demetriou, Dennis Sandow, Naomi Markovitz, Timothy Maloney, Dermott Breault, Jon Thaler, Curtis Brown, Harry Zaremba, Douglas Szper, Harry Hazard, Winslow Hartford, Gerald Blum, T. Mahon, Avi Ornstein, Jacob Bergmann, Lisa Chabot, Mike Bercher, Richard Hess, Michael Froman, Judith Longyear, and the proposer, William Schumacher.



MAY 5 A dog is lost in a square maze of corridors. At each intersection, he chooses a direction at random and proceeds to the next intersection or exits at one of the sides. His walk is over when he reaches one of the sides. What is the probability P_k that the dog, starting at intersection k, will exit at the south side?



The following solution is from Vic Elias, who was a graduate student in physics at Santa Cruz the year I taught mathematics there. His solution seems to me to look like a physicist's; he finds an extra symmetry most people miss. Of course, they solve the problem anyway, but that's beside the point. He writes:

The symmetry of the problem requires that $P_1 = P_2$, $P_3 = P_4$, and $(P_1 + P_2 + P_3 + P_4)/4 = 1/4$. [This last relation is equivalent to saying that if the dog is equally likely to be at any of the four intersections, his probability of exiting the maze in any given direction is 1/4]. Thus

$$2P_1 + 2P_3 = 1 \qquad (1)$$

Suppose the dog is at intersection 1. By letting it move one unit from intersection 1, we see that $P_1 = P_2/4 + P_3/4$, in which case

$$3P_1 = P_3. \qquad (2)$$

Combining (1) and (2),

$$P_1 = P_2 = 1/8 \text{ and}$$

$$P_3 = P_4 = 3/8. \qquad (3)$$

As a check, let the dog start at intersection 3. Then

$$P_3 = P_1/4 + P_4/4 + 1/4. \qquad (4)$$

The relations (3) are consistent with (4).

Also solved by Bob Kimble, John Pierce, Jeff McGuire, Steve Rosenberg, William Butler, Smith Turner, Paul Hertz, Peter McMenamin, Robert Bart, Jon Thaler, Winslow Hartford, Harry Zaremba, James Tanenbaum, Gerald Blum, Douglas Szper, Frank Carbin, T. Mahon, Richard Hess, Marshall Fritz, Rodney Weatherford, Judith Longyear, and the proposer, John Prussing.

1977 DEC 5 Irving Hopkins is not happy with Raymond Kinsey's solution. Mr. Hopkins' comments follow the drawing (see above) which shows his solution:

From my solution, given $(\theta + \phi) = 60°$,
$K = M/a = \cos 60° + (\cos^2 60° + 1)^{\frac{1}{2}} = 1.618034$
Given a = 8, M = 12.9442
$h = a(\sin 60°) = 6.928$
$\theta = \arctan (6.928/16.9442) = 22.2390°$
$\phi = 60° - \theta = 33.76°$
$\delta = 120°$
$\gamma = 180° - 120° = 60°$
$\alpha = \arctan (h/(M - 4)) = 37.76°$
$\beta = 180° - 60° - \gamma = 82.24°$.
From this, $\alpha = \theta + \phi = 60°$
$\delta = \alpha = \beta = 120°$,
not $(\gamma) = (\alpha + \beta)$ and $(\delta) = (\theta + \phi)$
$(60) = (120)$ and $(120) = (60)$

Also, from the above, $\beta = 82.24$, and $\theta = 22.24$, whence β cannot equal θ, as Mr. Kinsley claims. Hence the rest of the argument falls apart. Also, if we consider the simultaneous equations $R/S = M/L$ and $2R/L = M/S$, we can eliminate R and M, with the result that $L^2 = 2S^2$, which is obviously not generally true. Or similarly, by eliminating L and S, we find $M^2 = 2R^2$, equally untenable.

Proposer's Solutions to Speed Problems

SD 1 Sequoia.

SD 2 (courtesy of AG): Twelve plus one forty four; plus twenty plus three roots of four; divided by seven; plus five times eleven; gives nine squared and not a bit more.