Help Allan Solve His Cocktail Party Puzzle

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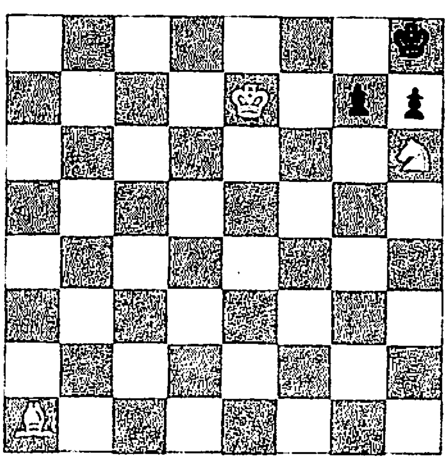
This is a sad time for Puzzle Corner. Our most active participant, R. Robinson Rowe (M.I.T. '18), died on May 4. In his last letter to "Puzzle Corner" dated April 17, Mr. Rowe mentioned that he was "lying on my back 98 per cent of the time." Harry Nelson, the Editor of the *Journal of Recreational Mathematics*, forwarded a copy of Mr. Rowe's letter to *JRM* dated May 1. The letter was concluded by Edwin R. Rowe who told us of his father's death. This issue of "Puzzle Corner" is dedicated to his memory.

A/S 1 We begin this month with a novel chess problem from Abe Schwartz. White is to play and mate in three moves. Nothing unusual about that, but you are to show that White can mate in three moving either up the board or down.



A/S 2 Michael Auerback submitted the same problem that a York student sprung on a few of our faculty. The faculty liked it; what do you think? (Incidentally, Mr. Auerback edits *The Valchemist*, the magazine of the Connecticut Valley Section of The American Chemical Society.)
Recently Allan Gottlieb and his wife attended a cocktail party at York College at which there were four other married couples. Various introductions and handshakes took place. No one shook hands with him- or herself, or with his or her spouse, and no one shook hands with the same person more than once. When Allan asked everyone how many people they had shaken hands with, to his surprise each person gave him a different answer. How many hands did his wife shake?

A/S 3 Eric Jamin sends us a geometry problem which was part of the 17th International Mathematics Olympiad:
Given any triangle ABC, construct outside it the three triangles BCP, CAQ, and ABR such that

$\angle$ PBC = $\angle$ CAW = 45°,
$\angle$ BCP = $\angle$ QCA = 30°, and
$\angle$ ABR = $\angle$ RAB = 15°.

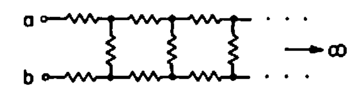
Show that $\angle$ QRP = 90° and that RQ = RP.

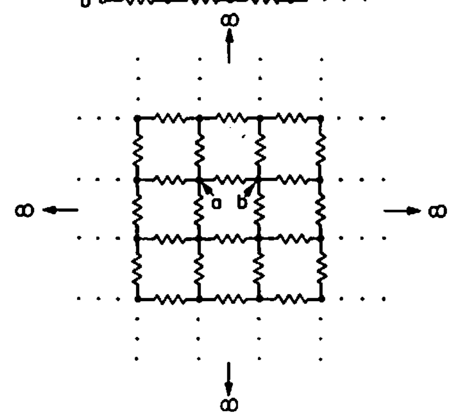
A/S 4 Consider the number 153. It turns out to be equal to the cubes of its digits — i.e., $153 = 1^3 + 5^3 + 3^3$. Find three other such numbers. This property can be extended to four-digit numbers — for example, $1634 = 1^4 + 6^4 + 3^4 + 4^4$. Find two other such four-digit numbers. In the same way, the property can be generalized for numbers of any size. Martin Gardner published a five-digit number of this type several years ago. Find a proof that there are an infinite number of such numbers, or else prove that only a finite number exist.

A/S 5 We close with a cryptarithmatic problem from Frank Rubin: Replace each letter by a unique decimal digit.

$$\begin{array}{r} \mathrm{H\ E} \\ \times\ \ \mathrm{I\ S} \\ \hline \mathrm{H\ U\ B} \\ \mathrm{O\ F}\ \ \\ \hline \mathrm{L\ A\ B} \end{array}$$

Speed Dept.
A/S SD1 and A/S SD2 Stephen Polloch claims that these two are speed problems in electrical engineering. Those of you who remember my exploits assembling hi-fi kits will understand that I am in no position to judge anything electrical. In both problems each resistor is 1 ohm and you are to find Rab.





Solutions
NS 11 Can any square matrix composed of just 0s and 1s have a determinant no greater than F_n (F for Fibbonacci), where F_n is defined by $F_1 = F_2 = 1$ and for n at least three $F_n = F_{n-1} + F_{n-2}$.

Jerry Griggs informs me that Herb Ryser has an article on this subject in the *Canadian Journal of Mathematics* 8 (1956) pp. 245-249. Although the exact upper bound is unknown, it is larger than Fn, as the following counterexample shows.

$$\begin{array}{ccccccc} 1&0&1&0&1&0&1 \\ 0&1&1&0&0&1&1 \\ 1&1&0&0&1&1&0 \\ 0&0&0&1&1&1&1 \\ 1&0&1&1&0&1&0 \\ 0&1&1&1&1&0&0 \\ 1&1&0&1&0&0&1 \end{array}$$

Harry Zaremba also responded and Richard Stanley submitted the following: Let f(n) be the largest determinant of an $n \times n$ matrix of +1s and −1s, and let g(n) be the largest determinant of an $n \times n$ matrix of 0s and 1s. If M is an $n \times n$ ± 1-matrix, then multiply rows and columns by ± 1 so that the first row and column contain all 1s. Now subtract the first row from the other rows and factor 2 out of all rows but the first. The last $n - 1$ rows and columns form a 0-1 matrix N with

$\det M = \pm 2^{n-1} \det N.$

This process of getting N from M can be reversed. It follows that

$g(n) = 2^{-n} f(n + 1).$

Hadamard's determinantal inequality

states that $f(n) \leq n^{n/2}$. Hence $g(n) \leq 2^{-n}(n+1)^{(n+1)/2}$. An $n \times n$ ± 1-matrix M is called a *Hadamard matrix* if

$$\det M = \pm n^{n/2}.$$

It is known that Hadamard matrices exist for infinitely many values of n — e.g., whenever n is a power of 2 or when $n - 1$ is a prime power and n is divisible by 4. (See M. Hall, Jr., *Combinatorial Theory*, p. 207). Hence, for infinitely many values of n,

$$g(n) = 2^{-n}(n+1)^{(n+1)/2}.$$

This is much larger than F_n. For any n one can show that g(n) is not much smaller than $2^{-n}(n+1)^{(n+1)/2}$.

It is interesting that even an "average" 0-1 matrix will have a large determinant (in absolute value). One can show that the average value of $(\det N)^2$, taken over all 2^{n^2} $n \times n$ 0-1 matrices N, is equal to $4^{-n}(n+1)!$ Thus there must be many N with

$$\det N \geq \sqrt{4^{-n}(n+1)!}$$

$$\sim \sqrt{2\pi}(2\sqrt{e})^{-n}\, n^{(n/2)+1},$$

which is much larger than F_n.

Your readers may be interested in trying to prove the following (difficult) result of Komlós. As $n \to \infty$, the percentage of $n \times n$ 0-1 matrices with determinant 0 approaches 0.

M/A 1 Given the following hands, and the bid of seven spades by South:

♠—
♥A 10 8 6
♦A K Q J
♣A J 10 7 5

West	East
♠9 2	♠7 6 5 4 3
♥Q 9 7	♥K 5 4 3 2
♦7 6 4 3 2	♦—
♣K Q 6	♣9 4 3

♠A K Q J 10 8
♥J
♦10 9 8 5
♣8 2

After the opening lead of the ♥Q, South has 12 tricks. Where will the 13th come from?

Many favorable comments about this problem were received. Also, many correct solutions. The following is from Frank Pollnow.

As in the case of many tricky bridge hands, the solution lies in "wasting" top honors. After taking the lead with the ♥A, declarer plays the ♥10 off the board, presumably East covers (or the 13th trick is thereby made) and South trumps with the ♠8. South then plays out the 5 trumps, discarding the four diamond honors in the dummy and a small club.

Next, South plays out the ♦10, ♦9, and ♦8, at which point West is hopelessly squeezed with only three cards remaining. If West has not retained a diamond and a heart, these tricks set up in the declarer's and dummy's hands, respectively, and together with the ♣A, secure the 13-trick contract.

If West retains a diamond and two clubs, the dummy's two remaining hearts comprise the 12th and 13th tricks. And if, of course, West retains one or less clubs, the contract is completed by cashing the three club honors on the board.

Also solved by Ronnie Rybstein, Glenn Brown, G. Holderness, Mike Bercher, Bo Jansen, John Lai and David Cochener, Steven Feldman, Rex Ingraham, Phyllis Grossberg, Mark Freundel, Jerry Grossman, Jacob Bergmann, John Rule, Edmund Chen, William Butler, Jerome Gordon, Edgar Rose, Douglas Van Patter, Gardner MacPherson, Sheldon Katz, Douglas Stark, Paul Horvitz, Steve Grant, James Shearer, David Olson, Michael Kay, Rudolph Evans, Edward Lynch, Eugene Biek, Emmet Duffy, and the proposer, Albert J. Fischer.

M/A 2 What is the smallest positive integer containing at least 1 million distinct proper factors? (The factors need not be prime; for example, 12 has four factors — 2, 3, 4, and 6.)

Many readers submitted solutions having the same order of magnitude (23) but two, Charles Rozier and Jacob Bergman, found answers a little smaller than anyone else. Those of you who enjoyed this one might try to find the smallest positive number with exactly one million proper factors. Mr. Rozier's solution follows:

As Gauss pointed out in *Disquisitiones Arithmeticae*, any integer can be factored into a unique set of n primes p_i with respective exponents e_i, and further, the number of distinct factors including 1 and the number N itself is given by

$$\prod_{i=1}^{n} (e_i + 1).$$

As implied by the example in the problem statement, 1 and N are not to count as factors among the required one million. So we seek

$$N = \prod_{i=1}^{n} p_i^{e_i}$$

where N is the minimum positive integer containing at least

$$F = \prod_{i=1}^{n} (e_i + 1) - 2 = 10^6 \text{ factors.} \qquad (1)$$

N will be minimum when log N is minimum. Consider minimizing log N as a function of any two exponents, say e_1 and e_2. If there is a pairwise minimum of log N, it will be found where the Jacobian is zero: .

$$\begin{vmatrix} \dfrac{\delta \log N}{\delta e_1} & \dfrac{\delta \log N}{\delta e_2} \\ \dfrac{\delta F}{\delta e_1} & \dfrac{\delta F}{\delta e_2} \end{vmatrix} = 0, \text{ or}$$

$$\begin{vmatrix} \log p_1 & \log p_2 \\ \dfrac{F}{(e_1 + 1)} & \dfrac{F}{(e_2 + 1)} \end{vmatrix} = 0.$$

This leads to the conclusion that for any pair of exponents e_i and e_j, the minimum of N corresponds to the condition

$$(e_j + 1) = (e_i + 1)\frac{\log p_i}{\log p_j}. \qquad (2)$$

If a value is selected for n, the number of prime factors of N, the approximate values of e_1 through e_n can be calculated from (1) and (2):

$$(e_i + 1)^n \approx (10^6 + 2)\frac{\prod_{j=1}^{n} \log p_j}{(\log p_i)^n}.$$

For example, if $n = 2$, the rounded values of e_1 and e_2 are 1258 and 793. But the rounding process in this instance results in a value for F less than 10^6, because $1259 \times 794 - 2 = 999644$. To increase F, either e_1 or e_2 could be increased by 1; however to minimize N it is expedient to decrease e_1 and increase e_2 by 1.

Then $e_1 = 1257$ and $e_2 = 794$, $F = 1{,}000{,}108$, and $N = 2^{1257} \cdot 3^{794} \approx 1.694 \cdot 10^{757}$.

Using the same procedure, the values of n, e_i ($i = 1, 2, 3 \ldots n$), N and F were determined and arranged in the tabulation at the top of the next column. It would appear from the table that N is a minimum for $n = 12$ or 13. However, again the round-off results in some values of F below 10^6. When these are adjusted upward, the minimum shifts to a higher value of n. Furthermore, some values of F are too large, so some of the e_i can be judiciously reduced to minimize N. The least value of N meeting the conditions that I could find by a few such adjustments was for $n = 15$, the last line (in bold type) in the table. Therefore I am submitting as my answer $N = 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47$, which has 1,013,758 factors; N = 297,508,272,408,041,611, 752,400.

Also solved by Peter Kerr, Glenn Brown, Jeffrey Wint, P. Lyons, R. Robinson Rowe, Sheldon Katz, Roger Milkman, Frank Carbin, Edward Lynch, Frank Rubin, Emmet Duffy, Doug Szper, and the proposer, Theodore Engle.

M/A 3 H. R. ("Tim") Lefever, farmer, environmentalist, and electrical engineer (M.I.T. '41), has an unforgettable phone number, whose digits can be represented as (abc) efg-wxyz. If anyone needs to telephone Lefever to inquire about his solar-heated house, built in 1954, one needs only to remember the following:

☐ The area-code (abc) is a palindrome the sum of whose digits is the square root of the exchange number (efg).

☐ The exchange number (efg) is the square of the sum of the last three digits of the phone number (wxyz).

☐ The last three digits of the phone number (xyz) are consecutive numbers, increasing order, and the sum of the cube of the first three equals the cube of the last.

Despite a minor typo in the problem statement, many readers were able to correctly solve this one. I presume that Mr. Lefever's phone has been quite busy. The following solution is from John Rule:

Consider wxyz; xyz are consecutive numbers such that $w^3 + x^3 + y^3 = z^3$, or $z^3 - (x^3 + y^3) = w^3$.

Hence we want three consecutive numbers such that the cube of the largest minus the cubes of the two smallest yields a number which is itself a cube. A short inspection shows that there is only one such combination: $6^3 - (5^3 + 4^3) = 3^3$. Hence wxyz = 3456; efg = $(4 + 5 + 6)^2 = 225$; and the sum of the digits of abc = 15.

My definition of a palindrome is that it is any number, word, sentence, or what have you, that reads the same forwards as backwards. Unless there is some further restriction of which I am unaware there are five palindromes whose digits add to fifteen: 393, 474, 555, 636, and 717. To get out of this dilemma I must resort to the fact that all the area codes the good A.T.&T. has yet assigned have either a 1 or a 0 as their central digit. This leaves only 717, so I must infer that Mr. Lefever's telephone number is 717-225-3456 and that he lives on a farm in Pennsylvania.

Also solved by: Peter Kerr, Christopher

n	e_i	N	F
1	1000001	1.98E301030	1,000,000
2	1258, 793	1.13E757	999,644
3	153, 96, 65	1.96E137	985,908
4	56, 35, 23, 19	4.90E65	984,948
5	31, 19, 13, 11, 8	1.29E45	967,678
6	22, 13, 9, 7, 6, 5	7.07E36	1,081,918
7	17, 10, 7, 5, 4, 4, 3	7.54E33	2,851,198
8	14, 9, 5, 4, 3, 3, 3, 2	1.25E28	863,998
9	12, 7, 5, 4, 3, 3, 2, 2, 2	1.08E28	1,347,838
10	11, 6, 4, 3, 2, 2, 2, 2, 2, 1	1.05E25	816,478
11	10, 6, 4, 3, 2, 2, 2, 2, 1, 1, 1	7.06E24	997,918
12	9, 5, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1	4.58E23	829,438
13	9, 5, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1	1.58E23	829,438
14	8, 5, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1	3.39E24	1,492,990
15	8, 5, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1	1.23E25	1,990,654
16	8, 5, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1	5.91E25	2,654,206
17	8, 4, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1	1.16E27	4,423,678
15	**10, 4, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1**	**2.975E23**	**1,013,758**

Roth, Ronnie Rybstein, Jeffrey Wint, Glenn Brown, Mike Bercher, Harry Zaremba, Bill Taylor, Winthrop Leeds, P. Jung, Nancy Hall, Elizabeth Sawyer-Klaeson, Rex Ingraham, William Butler, R. Robinson Rowe, P. Lyons, Johan Norvik, Steve Grant, Roger Milkman, Gardner Perry, Patsy Henry, James Sheare, Edward Lynch, John Prussing, Turner Gilman, Naomi Markovitz, Frank Rubin, Doug Szper, Dawn Swift, and the proposer, Neil Hopkins.

M/A 4 Consider a solitaire card game (called accordion, among other names, which consists of dealing a deck, one card at a time, and then examining sets of four cards. If the four cards are of the same suit, the middle two are discarded. If the four cards are of the same value, all four are discarded. What are the odds of winning (no cards left)? What if the whole deck is laid out before starting?

No one solved this completely. P. Lyons feels that if the game is restricted to one pass through the deck the probability of success is

$$\frac{2^{42}}{51! - 13!}$$

Jerrold Grossman offered the following comments:

As described, the game is almost impossible to win, although an analytic answer to the first question posed is probably possible. In the version with which I am familiar, the four cards are removed if the first and the fourth are of the same value, while the middle two are discarded if the first and the fourth are of the same suit. Now an analytic solution is next to impossible, but I offer the following data from a computer simulation of 10,000 games, requiring 22 minutes on a Burroughs 5500. Assuming my random number generator "worked," the probability of winning is around .0054 with a standard error of estimation of .0007. Actually the median number of cards remaining at the end of the game turned out to be 14; the mode was 12 (occurring 1014 times); the mean was 14.22 and the standard deviation was 7.59. In the worst game there were 42 cards left at the end.

M/A 5 It is well known that the trigonometric functions of certain spiral angles are algebraic numbers. For example, $\sin 45° = \sqrt{2}/2$, $\sin 15° = \frac{1}{4}(\sqrt{6} - \sqrt{2})$. What is the smallest integer angle A for which such an explicit closed-form expression for sin A may be obtained?

We should have been more precise. By "closed form" we meant "solvable by radicals" or "expressible in surds." It is fitting that our last solution for academic 1977-1978 comes from R. Robinson Rowe:

I note first that algebraics for sin, cos and, tan of multiples of 15° and 18° have been published (e.g., Carr's *Synopsis of Pure Mathematics*). These can be extended by formulae for sines of half angles and difference of two angles. Thus:

$$\sin 3° = \sin (18° - 15°) =$$
$$(\sqrt{5} - 1)/4 \cdot (\sqrt{3} + 1)/2\sqrt{2}$$
$$- (\sqrt{3} - 1)/2\sqrt{2} \cdot \sqrt{5 + \sqrt{5}/2\sqrt{2}}$$

in which I have preserved the expressions for functions of 15° and 18°. Little simplification results from multiplying it out. We cannot get down to 2° or 1° without trisecting an angle. The geometric problem is known to be impossible.

Algebraically, it requires the solution of a cubic with three real roots, which is known to be impossible in surds. Hence the answer is 3°, and the result can be stated more generally: if A is an angle of an integral number of degrees, its sine can be expressed algebraically if and only if A is divisible by 3.

Also solved by: William Butler, Sidney Kravitz, Sheldon Katz, Harry Zaremba, Norman Megill, Naomi Markovitz, Frank Rubin, and the proposer, Stephen Hirshman.

Better Late Than Never

1977 DEC 2 John Allen submits the following:

Please note that there is an additional discrepancy between Mr. Shearer's and Mr. Hartford's solutions to the DEC 2 problem not mentioned in the discussion in the March/April issue. Mr. Shearer assumes that the piston has no mass but that the gas has mass. Otherwise, the dissipative processes (wave motions and shocks) which he mentions would not occur — not as mechanical processes, in any case. If one were to consider electromagnetic radiation confined within perfectly conducting walls and its radiation pressure, wave motion would be in the nature of things, but shocks would not be possible. However, there would be no dissipation within the medium, only within the walls — that is, if the walls were *not* perfectly conducting. This is an example of a case in which the "gas" is indeed massless. Mr. Hartford makes the distinction between a piston with and without mass. But his conditions for a solution given a piston without mass are still not sufficient, because he does not say whether or not the gas has mass. If it does, energy can indeed be stored in the gas itself, and the process will not be instantaneous. There will be oscillations within the gas at both ends of the piston. Consequently, the piston, too, will oscillate. The same is true, of course, even if the piston does have mass.

The conclusion I reach is: science is objective. I think this has been said before — the solution depends on how many factors you are willing and able to bring to bear in analyzing the problem. No solution given so far to the DEC 2 problem is exhaustive, and none, in fact, ever will be. We can set artificial conditions, like perfectly insulating walls, to simplify the problem, but even so, difficulties creep in.

DEC 3 Raymond Cowen and Alan Walter have responded.

Dec 4 R. Boas, the editor of the *American Mathematical Monthly,* notes that a paper on this subject by H. Gould will appear in the June-July issue.

DEC 5 John Rule noted that two quantities were omitted from the diagram:

$L = \overline{CB}$ and $S = \overline{PB}$.

Additional responses have been received: FEB 1, FEB 2, and FEB 3 Frank Rubin.

FEB 4 Frank Rubin, Warren Lane, and R. Robinson Rowe.

FEB 5 Frank Rubin.

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