Adventures on a Leisurely Trip from Aardvark to Zymurgy

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Your editor is very pleased to report that he is now an (unpaid) television "star." In keeping with my high journalistic standards (I "allow" "Puzzle Corner" to appear only in *Technology Review*), my television debut (and finale) was with the best show on television, "Nova." The crew from WGBH came to the World Computer Chess Championship to film a segment for "The Mind Machines," an account of artificial intelligence. I was there, too, and for a fleeting instant our paths crossed.

Enough said; let's get down to business.

Our backlogs are all quite large except for chess problems, of which there is a near-critical shortage.

Problems

JUN/JUL 1 We start with a bridge problem from William Butler:

♠ 7 3
♥ Q J
♦ 5 2
♣ A K 5 3 2

♠ A Q 2
♥ A K 7 6 5 4
♦ A Q
♣ J 6

With South the declarer at six hearts, West leads the ♥10. South thinks a moment or two and plays a heart from dummy. East discards a diamond. What is the best play to make six hearts? If possible, supply the probability of success.

JUN/JUL 2 Our second problem, from Frank Rubin, concerns a leisurely effort to avoid colliding with a train:

I am walking the 25 miles from Aardvark to Zymurgy. Part of the way there, I begin to cross a railroad bridge. When I have gone three miles along the bridge, I hear a diesel train leaving the Aardvark station. I could exactly escape the train by running to either end of the bridge. But instead I cross over to the other track and walk four miles further. Now I see a steam train leaving the Zymurgy station on my track. Again, I could exactly escape by running to either end of the bridge. But instead, since the diesel train has just passed, I switch back to the first track. I reach the end of the bridge just as the steam train reaches the start of the bridge (in my direction). One hour later, I reach Zymurgy. How long is the bridge?

JUN/JUL 3 Eugene Sard wants you to solve the following pair of equations for x and y:

$$1 - xy = x + y^2$$

$$1 - xy = y + x^2$$

JUN/JUL 4 Jeff Kenton must be a lonely man. He has invited all *Technology Review* readers to call him at home. Unfortunately, all he will tell us about his telephone number is that it has the following property:

$$myphone = (my)^{*}(hy)^{*}(p)^{4}$$

Each letter represents a different digit (none repeats), and neither of the first two digits can be 0 or 1 (it's a telephone number, remember).

JUN/JUL 5 John Gray asks us a geometric construction problem:
Given a line and two points on one side of the line, construct the smaller circle which passes through the points and is tangent to the line.

Speed Department

JUN/JUL SD 1 Harvey Elentuck knows some pairs of "crazy fractions" — ones that add the same way as poor algebra students. The first type satisfy:

$$a/b + c/d = (a + c)/(b + d).$$

For example, $-4/12 + 1/6 = -3/18$, and $25/10 + -81/18 = -56/28$.
The second type satisfy

$$a/b - c/d = (a - c)/(b - d).$$

For example, $15/10 - 16/8 = -1/2$.

What conditions on a, b, c, and d guarantee a pair of the first type? The second type?

JUN/JUL SD 2 Winthrop Leeds has a "non-algebraic" way to solve the following problem:
A man starts rowing steadily upstream from point A on a certain river. At point B exactly one mile upstream he notices an odd-shaped log floating by. He continues to row for an hour more before turning around and rowing back downstream. When he arrives back at his starting point A, he observes that he has just caught up with the floating log. What is the velocity of the river?

Solutions

FEB 1 Place one White King, two White Rooks, and one Black King so that White, who is to move, can mate with any of four moves.

Hal Moeller supplied us with the following solution:
White King at K1 (unmoved), White Rook at KR1 (unmoved), White Rook at QB2, and Black King at KR8. White's four mating moves are K to Q2, K to K2, K to KB2, and castles.

Also solved by Lindsay Faunt, Steven Ross, Peter Siczenicz, Jay Anderson, William Butler, Eric Piehl, S. D. Turner, Abe Schwartz, James Shearer, and the proposer, Steven Grant.

FEB 2 When does $[\sqrt{n}]$ divide n where [ ] is the floor or greatest integer function? More generally, when does $[{}^{*k}\sqrt{n}]$ divide n?

The following solution is from Naomi Markovitz:

If n is a perfect square, obviously $[\sqrt{n}]$ divides n. Otherwise we want ab = n:

$$a < \sqrt{n} < (a + 1)$$

$$a^2 < n < (a + 1)^2$$

$a < n/a < (a + 2 + 1/a)$. This last is

$a < b < (a + 2 + 1/a)$. Hence b is no more than 2 greater than a. Thus n can be expressed as $a(a + 1)$ or $a(a + 2)$ as well as a^2. In the general case we want ab = n:

$$a \le n < (a + 1)$$

$$a^k \le n < (a + 1)^k$$

$$a^{k-1} \le n/a < (a + 1)^k/a$$

$$a^{k-1} \le b < (a + 1)^k/a.$$

Also solved by William Butler, Winslow Hartford, and Jerry Griggs.

FEB 3 Given an n-by-n checkerboard and n^2 checkers of n different colors, and given that there are n checkers of each color, is it possible to arrange all the n^2 checkers on the board so that no two checkers of the

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

1	2	3	4	5	6	7
3	4	5	6	7	1	2
5	6	7	1	2	3	4
7	1	2	3	4	5	6
2	3	4	5	6	7	1
4	5	6	7	1	2	3
6	7	1	2	3	4	5

same color lie in the same row, column, or diagonal? (By diagonal is meant *all* the diagonals, not just the two main diagonals.)

Everyone agrees that this is impossible for n even. Harry Zaremba gives us an algorithm for two-thirds of the odd integers; since by exhaustive search one can show that no solution is possible for n = 3 (he predicts this), his algorithm may be the best possible. My feeling is that the problem remains not completely solved (n even and one-third of n odd) and may appear as an NS-problem in the 1980s. (An interesting variant of this problem has just been posed by Paul Monsky, a professor of mine at Brandeis, in the American Mathematical Monthly.) Mr. Zaremba's comments follow:

Solutions are possible for all odd values of n such that $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. Arrangements for n = 5 and n = 7 are shown below, in which the different tones are denoted by distinct integers. Solutions for even values of n are not possible. Arrangements can be developed easily by means of the following procedure:

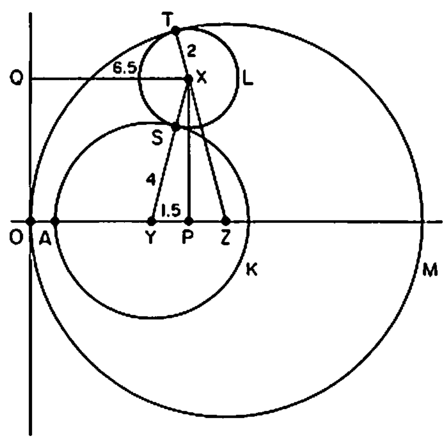
1. In the first column, place the consecutive odd integers in order followed by the consecutive even integers.
2. Arrange the integers in consecutive order in each row starting from the integer in the first column progressively to the largest integer, and resume the consecutive order with the remaining integers.

Many other patterns are possible. The sequence of integers in the main diagonal downward to the right offers a clue to why odd values of n divisible by 3 cannot be arranged to meet the requirements.

Also solved by Jay Anderson, Eric Piehl, Abe Schwartz, and the proposer, Sheldon Razin.

FEB 4 In the figure, circle L is tangent to circles K and M, and the shortest distance from its center to the tangent at 0 is 6.5 inches. If the distance between tangents at 0 and A is 1 inch, and the radii of circles K and L are 4 and 2 inches, respectively, what is the radius of circle M? What is the locus of the centers of all circles which are tangent to both circles K and M?

The following solution is from Shirley Wilson:



Let X, Y, and Z be the centers of circles L, K, and M, respectively. Let S and T be the points where circle L is tangent to circles K and M, respectively. Let P and Q be the projections of X onto the diameter of M and the tangent at 0, respectively.

Since $|QX| = 6.5$, $|OA| = 1$, $|AY| = 4$, and $|OP| = |QX|$, $|YP| = 1.5$. The segment XY is between the centers of L and K, and hence, $|XY| = 2 + 4 = 6$. Applying the Pythagorean Theorem to ΔXPY, $|PX|^2 = 36 - 2.25 = 33.75$.

The radius of M is $|TZ| = |TX| + |XZ| = 2 + |XZ|$. The radius of M is also $|OZ| = |OP| + |PZ| = 6.5 + |PZ|$. Therefore, $2 + |XZ| = 6.5 + |PZ|$, or $|XZ| = 4.5 + |PZ|$.

Using ΔXPZ, $|PX|^2 + |PZ|^2 = |XZ|^2$. Substituting, one finds that $|PZ| = 1.5$. Hence the radius of M is 6.5 + 1.5 = 8.

Now let L be a circle tangent to circles K and M such that the radius of L is r. Using the notation above, $|XY| + |XZ| = (4 + r) + (8 - r) = 12$. Hence, X traces an ellipse with foci at Y and Z, the centers of K and M, and with a major axis of length 12. If 0 is the origin of a coordinate system, the foci are Y = (5,0) and Z = (8,0), so that the center of the ellipse is at P = (6.5,0). The minor axis is of length $2(33.75)^{1/2}$ since in the previous problem, PX was perpendicular to OP and $|PX|^2 = 33.75$. Hence, the equation of the ellipse would be

$$(x - 6.5)^2/36 + y^2/33.75 = 1.$$

Also solved by Jay Anderson, William Butler, Calvin Simmons, Naomi Markovitz, Winslow Hartford, W. C. Kenner, John Rule, Sheldon Katz, Harold Heins, Norman Wickstrand, Emmet Duffy, James Shearer, and the proposer, Harry Zaremba.

FEB 5 "Tick Tack Math" has 52 playing cards numbered 1 to 40 with 1 to 12 repeated. The mechanics of the game involve using two-card mathematical combinations to equal upturned single cards from the same deck. Addition, subtraction, division, and multiplication are all acceptable combinations. A single card that matches an upturned card can also be played. The question is: Ignoring knowledge of other upturned, played, or held cards, how many practical ways (including permutations) are there of making each number? "Practical" is meant to eliminate from the solution such combinations as 2 − 1 = 1 which, while mathematically correct, waste a card.

There seems to be some controversy as to whether or not to count 13 − 12 as two solutions (using the two different 12s in the deck) or as one solution. My guess is that the former was intended, but I am not sure; the proposer, Richard J. Alden, agrees with me. Of course, this would mean that 5 − 4 yields four solutions. Avi Ornstein holds the view that 13 − 12 is only one solution, and his answer is shown in the box on the next page. Naomi Markovitz has the satisfaction of knowing that she agrees with me. However, she neglected to consider the possibility of solutions like 4 which involve only one number (i.e., no arithmetic operators). No one considered the possibility of more than one operator — a good thing (see PERM 2 for an idea of what can happen). Anyone desiring a copy of "Tick Tack Math" should write to Richard J. Alden, 1182 Sesame, Sunnyvale, Calif. 94087.

Value	Identity	Addition	Subtraction	Multiplication	Division	Total
1	1		38		11	50
2	1	1	37		18	57
3	1	1	36		11	49
4	1	2	35	1	8	47
5	1	2	34		6	43
6	1	3	33	1	4	42
7	1	3	32		4	40
8	1	4	31	1	4	41
9	1	4	30	1	3	39
10	1	5	29	1	3	39
11	1	5	28		2	36
12	1	6	27	2	2	38
13		6	26		2	34
14		7	25	1	1	34
15		7	24	1	1	33
16		8	23	2	1	34
17		8	22		1	31
18		9	21	2	1	33
19		9	20		1	30
20		10	19	2	1	32
21		10	19	1		30
22		11	18	1		30
23		11	17			28
24		12	16	3		31
25		12	15	1		28
26		12	14	1		27
27		13	13	1		27
28		13	12	2		27
29		14	11			25
30		14	10	2		26
31		15	9			24
32		15	8	2		25
33		16	7	1		24
34		16	6	1		23
35		17	5	1		23
36		17	4	4		25
37		18	3			21
38		18	2	1		21
39		19	1			20
40		19		3		22
						1,289

Better Late Than Never

Y1977 Alan Katzenstein and Irene Greif correct two misprints, namely

22 = [(9 − 1) + 7] + 7
81 = 9**(1 + 7/7).

They also give two improvements which have the digits in the preferred order and still require the same number of operators as the published solutions:
16 = (1 + 9/7)*7
50 = (1**9) + (7*7).

1977 O/N2 Eric Jamin was the proposer; Emmet Duffy and John Rule have responded.

1978 JAN 3 Ely Stern, Thomas Sico, Bob Franzosa, Woodruff Sullivan, and the team of Chaplick Chaplick McDonough and Lennox have responded.

JAN 4 John Prussing has responded.

PERM 2 Up to 130, we need solutions for 87, 93, and 107. The object is to create these numbers using exactly four 4s and the minimal number of arithmetic operators — ! (factorial), √ (square root), and . (decimal point). I have received responses from C. Little, G. Ropes, H. Goldman, M. Gasser, H. Zaremba, F. Rubin, and H. Hazard. Two improvements and some new answers (up to 170) follow; more will be given in December.

Let $Z = \sqrt{\sqrt{\sqrt{.\sqrt{4}**(-4!)}}} = 125$

$X = \sqrt{\sqrt{\sqrt{4**(4!)}}} = 64$

106 = (44/.4)−4
115 = (44 + √4)/.4
134 = 44/.4 + 4!
135 = Z + 4/.4
136 = (4! + 4)*4 + 4!
137 = Z + 4!/√4
138 = (4!*4! − 4!)/4
139 = (4! + 4 − .√4)/(.√4)
140 = [(4!)/.4 − 4]/.4
141 = Z + 4*4
142 = 4!*4!/4 − √4
143 = (4!*4! − 4)/4
144 = (4 + √4)! − 4!*4!
145 = (4!/.4 − √4)/.4
146 = (4!/.4)/.4 − 4
147 = Z + 4! − √4
148 = (4!/.4)/.4 − √4
149 = (4!/.4 − .4)/.4
150 = √(4*4!/.4/.4)
151 = (4!/.4 + .4)/.4
152 = 44*4 − 4!
153 = Z + 4! + 4
154 = (4!/.4)/.4 + 4
155 = (4!/.4 + √4)/.4
156 = (4 + √4)!/4 − 4!
157 =
158 = (X/.4) − √4
159 = (X − .4)/.4
160 = 4**4 − 4!*4
161 = (X + .4)/.4
162 = X/.4 + √4
163 =
164 = 44 + (√4/.4)!
165 = (X + √4)/.4
166 =
167 =
168 = (4!*4!)/4 + 4!
169 = √(4! + √4)/√(4**4)
170 = (44 + 4!)/.4

Proposers' Solutions to Speed Problems

JUNE SD 1 $b^2/d^2 = a/c$; $d^2/(2bd - b^2) = c/a$.

JUNE SD 2 Instead of following a fairly straightforward approach of setting up the critical equation, from which the rowing speed s drops out, the following mental reasoning will provide the answer: From B the man rows upstream in one hour the distance (s − r) miles, where r is the speed of the river. From turnaround the man rows downstream *in one hour* (s + r) miles. The difference is 2r miles, which is exactly the distance the log has floated in the two hours since it was first sighted at B. Thus the man has caught up with the log. To make this catch-up point one mile below B at A, 2r = 1; and r = ½ mi./hr.