"Diablo Hand" and a Solar Hot Line

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One of our readers who shall remain nameless has actually expressed an interest in compiling an anthology of selected problems from Puzzle Corner. Considering the author I expect that the book will be quite good; but I wonder to whom we should give the movie rights. Any suggestions on problems to include/omit will be forwarded to the author.

I am always amazed by the geographical dispersion of my readers. Receiving a letter from Tony Osmond, Science Editor of *The Sunday Times Magazine*, was certainly a pleasure. Contrary to the beliefs of many New Yorkers, *The Times* is published in merry old London.

R. Robinson Rowe suggests another way of looking at my comment that York College freshmen look younger each year: "Ipso facto, as a corollary," he writes, "you look more mature and professional to the incoming freshmen. More at Harvard than at M.I.T., I noted and remember how the recruits to academe, as teaching fellows, instructors, and even assistant professors, feigned maturity with dignified reserve, pretense of sophistication, and sprouting hirsuteness (a van dyke or mutton-chops). But the old masters, the full professors and occupants of Gordon McKay chairs (as a rule but with exceptions), were clean-shaven, gregarious, sociable companions of students — in classroom and stadium. There was a legend of Charley and Ed en route via horse and buggy to the Harvard-Yale football game. Charlie was President Charles W. Eliot; Ed was Edward Everett Hale. As they drove slowly through Harvard Square, a friend asked, 'Where ya going, Charley?', which was answered by 'To yell with Hale!' "

One item of business: chess problems are in very short supply.

Problems

M/A 1 We begin with a bridge problem from Albert J. Fischer — a hand which he says is entitled the "Diablo Hand" in Albert Ostrow's long-out-of-print *The Bridge Player's Bedside Companion*. Having set it down from memory, Mr. Fischer warns that the "spot cards may not be exact, but the ones that matter are properly placed."

Given the following hands, and the bid of seven spades by South:

	♠ —	
	♥ A 10 8 6	
	♦ A K Q J	
	♣ A J 10 7 5	
♠ 9 2		♠ 7 6 5 4 3
♥ Q 9 7		♥ K 5 4 3 2
♦ 7 6 4 3 2		♦ —
♣ K Q 6		♣ 9 4 3
	♠ A K Q J 10 8	
	♥ J	
	♦ 10 9 8 5	
	♣ 8 2	

After the opening lead of the ♥Q, South has 12 tricks. Where will the 13th come from?

M/A 2 Theodore Engle believes that he knows the smallest positive integer containing at least 1 million distinct proper factors. What is this factor champion? (The factors need not be prime; for example, 12 has four factors — 2, 3, 4, and 6.)

M/A 3 Neil Hopkins has an easy way to remember Tim Lefever's phone number: H. R. ("Tim") Lefever, farmer, environmentalist, and electrical engineer (M.I.T. '41), has an unforgettable phone number, whose digits can be represented as (abc) efg-wxyz. If anyone needs to telephone Lefever to inquire about his solar-heated house, built in 1954, one needs only to remember the following:

☐ The area-code (abc) is a palindrome the sum of whose digits is the square root of the exchange number (efg).

☐ The exchange number (efg) is the square of the sum of the last three digits of the phone number (wxyz).

☐ The last three digits of the phone number (xyz) are consecutive numbers, increasing order, and the sum of the cube of the first three equals the cube of the last. ("I hope I have been helpful," writes Mr. Hopkins.)

M/A 4 Frederick Kummer has submitted the following:

My favorite solitaire card game (called accordion, among other names) consists of dealing a deck, one card at a time, and then examining sets of four cards. If the four cards are of the same suit, the middle two are discarded. If the four cards are of the same value, all four are discarded. What are the odds of winning (no cards left)? What if the whole deck is laid out before starting?

(Mr. Kummer notes that he is not aware of a solution, and he wonders if one is obtainable "without serious number-crunching.")

M/A 5 Stephen Hirshman has a trigonometry problem for us:

It is well known that the trigonometric functions of certain spiral angles are algebraic numbers. For example, sin 45° = $\sqrt{2}/2$, sin 15° = $\frac{1}{4}(\sqrt{6} - \sqrt{2})$. What is the smallest integer angle A for which such an explicit closed-form expression for sin A may be obtained?

Speed Department

M/A SD 1 James Cawse and Peter Stonestrom offer us a problem that was proposed and solved by a group of mountaineering mathematicians at the top of the talus pile below Tapeats Cave, Grand Canyon: Do the number of possible routes down a talus pile form a countable or uncountable infinity?

M/A SD 2 We close with a problem from John E. Prussing:

Consider a one-dimensional pursuit-evasion problem in which, at the initial time, the evader is a distance x ahead of the pursuer. The pursuer and evader travel at constant speeds p and e, respectively. In terms of $\alpha = e/p$, derive a simple expression for x^*, the position at capture (assuming the pursuer starts at the origin) in terms of x_0 and α, and a simple expression for the time of capture t^* in terms of t_0 and α (where t_0 is the time required for the pursuer to reach the initial location of the evader). What is the condition on α for capture?

Solutions

DEC 1 White to play and mate in two; see chessboard on facing page:

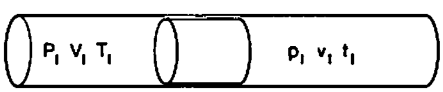
As I anticipated, this problem caused little trouble. Michael Delaney polished it off as follows:

White must begin with B — B2. The continuing variations are:

Black:	*White:*
N–Q2 or N–B3	B–N6 mate
N–B5 or N–B6 or N–N5	B–N6 or B–R4 mate
N–Q6 or N–B2	B–R4 mate
N–N3	B × N mate.

Also solved by E. Leroy, Robert Lack, Jeffrey Wint, Avi Ornstein, J. Fuss, Winthrop Leeds, Lindsay Faunt, Raymond Kinsley, Jacob Bergmann, William Butler, Jr., R. Robinson Rowe, Frederick Bercher, Ken Haruta, Rudy Evans, Scott Byron, Alan Peaceman, Roger Milkman, Gerald Blum, James Shearer, Harry Zaremba, Edward Lynch, Abraham Schwartz, and the proposer, Glen Ferri.

DEC 2 The ends of a closed cylinder, fitted with a leakproof, frictionless piston, are filled with perfect gases having the initial pressures, volumes, and temperatures indicated. If all of the walls are perfect heat insulators, where will the piston finally stop? Three students propose three answers: A says it will stop where P = p, using adiabatic processes. B says the piston will oscillate perpetually. C says: even though heat does not flow *through* the piston, the piston itself will act like a big molecule, and (after many oscillations) the pressures *and temperatures* will equalize. Who is right? Other possibilities?



In marked contrast to the previous problem, no two correspondents agreed! I have selected two views for inclusion and await the slings and arrows I fear will result. Actually, it seems to me that James Shearer and Winslow Hartford are not in disagreement but rather make different assumptions. Mr. Shearer writes:

I do not agree with any of the three students. A is wrong because adiabatic processes are not dissipative, so the piston will never stop if adiabatic processes are assumed. B is wrong because no compression or rarefaction, however slow, ever becomes *exactly* adiabatic; there is always some dissipation, so eventually the piston will stop. C is wrong about the temperature because the definition of equal temperatures implies either thermal contact, which the piston does not supply, or thermal contact with a third body. I say that the piston will eventually stop due to dissipative processes in the two gases (wave motions, or even shocks at high enough velocities). The two pressures will then be equal, but not the two temperatures.

Mr. Hartford writes:

The piston is not adequately defined. If it's *weightless*, as is usual in problems of this type, then energy cannot be stored, and assuming $P_1 > p_1$ will expand adiabatically, while the gas at p_2 will be compressed until $P_e = p_e$; the process will be instantaneous, and $T_e \neq t_e$, in general.

If the piston has *mass*, then the loss in energy will take place as a finite process, and the piston will move into the space of the gas originally at p_1 until $p_2 > P_2$; it will then oscillate back and forth with diminishing amplitude until $P_e = p_e$; the process remains adiabatic and the magnitude and period of the oscillations will depend on the mass of the piston.

If the piston is *not* an insulator, then heat will flow between the two chambers until $T_e = t_e$, and the pattern observed will depend on the thermal conductivity factor of the piston. However, the two final equilibria will depend on the insulating nature of the piston. Solutions follow:

Piston insulating:

Chamber A (left): P_1, V_1, T_1; moles of gases $N = P_1V_1/RT_1$.

Chamber B (right): p_1, v_1, t_1; moles of gases $n = p_1v_1/Rt_1$.

Equilibrium: $P = P_e = p_e$; in chamber A: $N^{RT}{}_e/V_e$; in chamber B: $n^{Rt}{}_e/v_e$.

Since $V_1 + v_1 = V_e + v_e$, it is possible to solve for T_e and t_e, if we bear in mind that $N(T_1 - T_e) = n(t_e - t_1)$ and also know the heat capacity ratios of the ideal gases; for adiabatic expansion $P_1V_1^K = P_2V_2^K$, where K = ratio of C_p to C_v; for an ideal monatomic gas, K = 5/3. To give an example, let P_1, V_1, T_1 = 2 atm., 2 liters, 400° K., and p_1, v_1, t_1 = 1 atm., 1 liter, 300° K. The equilibrium conditions are:

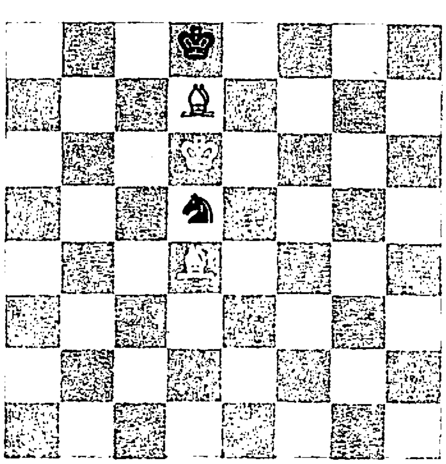
P_e = 1.6667 atm. p_e = 1.6667 atm.
T_e = 375.97° K t_e = 372.10° K
V_e = 2.2558 liters, v_e = 0.7442 liters.

If the piston is *thermally conducting*, then both P_e and p_e, and T_e and t_e are equal and the problem is readily solved by determining the volume occupied by the total gas, and the T as simple mixing of the moles of gas. Assume the same starting conditions; now

P = 1.6667 atm., as before
T = 375° K
V_e = 2.250 liters, v_e = 0.750 liters.

If the piston has mass, then a plot of V, v and T, t, and P, p versus time and a complex calculus operation are needed.

Also solved by: R. Robinson Rowe, William Butler, Jr., Gerald Blum, Harry Zaremba, and Edward Lynch.



DEC 3 Replace each letter by a unique digit to obtain a valid addition:

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   F I V E
     T W O
 +   O N E
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 E I G H T
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I have pooled the responses from Theodore R. Goodman and Avi Ornstein:

1. E must be 1.
2. F is either 8 or 9, since the greatest sum of three different digits is 24 and F plus the carried number must total at least 10.
3. I is either 0 or 1, since the sum of F and the carried number comes to 10 or 11.
4. Since E = 1, I = 0.
5. Since I = 0, the greatest sum in the hundreds column is less than 20; so F = 9.
6. Based on the ones column, T is 2 greater than O.
7. Since O + T produces G and carries over one number in the hundreds column, O is 5 or 6 and T is 7 or 8.
8. G is 2 or 4 if nothing is carried from the tens column, 3 or 5 if 1 is carried from the tens column, and 4 (since O and G can't both be 6) if 2 is carried from the tens column.
9. In both the first and third cases, the remaining digits for V, W, N, and H include an odd number of odd digits, which cannot fit into the addition.
10. Only eight possible combinations exist. It is easy to find that only one works: 2 + 4 + 7 = 13.
11. Thus O = 6, T = 8, G = 5, H = 3, and V, W, and N are 2, 4, and 7, respectively. For the last three letters, any of the six possible permutations will work. This means that there are actually six distinct solutions, namely:

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 9021     9021     9041
  846      876      826
  671      641      671
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10538    10538    10538

 9041     9071     9071
  876      826      846
  621      641      621
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10538    10538    10538
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Also solved by E. Leroy, Robert Lack, Lindsay Faunt, Mary Shooshan, Charles Mahlmann, James Cooney, Theodore Goodman, Ronald Newman, Raymond Kinsley, Mike Bercher, Scott Byron, Karen Larsen, Jules Sandock, R. Robinson Rowe, William Butler, Jr., Robin Smith, Alan Schwartz, Doug Patz, Sol Hekier, Bob Lutton, Jacob Bergmann, Jon Warren, Doug Szper, Alan Peaceman, Gerald Blum, Harry Zaremba, James Shearer, Edward Lynch, Winslow Hartford, M. Dow, Harry Zantopulas, Winthrop Leeds, and Abraham Schwartz.

DEC 4 It is well known and easily proved that the differences between consecutive perfect squares are always odd numbers and that the difference between two consecutive differences will always be 2, which equals 2!. It is also true that the differences of the differences of the differences of consecutive perfect cubes is always 6 = 3!. In fact, this pattern holds for all natural numbers and zero; i.e., (the differences of)^n consecutive perfect nth powers is n!. Below are several arrays which attempt to demonstrate this more clearly. The bottom line consists of consecutive integers all raised to the same power. Each higher line consists of numbers each the difference of the two numbers beneath it. Note that after making the same number of subtractions as the power in the bottom row, we obtain a row all of whose elements are the factorial of that power. Prove that this is so.

	2	2	2	2	2		2! = 2
1	3	5	7	9	11		
0	1	4	9	16	25	36	

		6	6	6	6		3! = 6
	6	12	18	24	30		
1	7	19	37	61	91		
0	1	8	27	64	125	215	

			24	24	24		4! = 24
		36	60	84	108		
	14	50	110	194	302		
1	15	65	175	369	671		
0	1	16	81	256	625	1296	

The following is from Doug Szper:

In standard functional notation, this problem can be stated as:

To prove: $\Delta^n x^n = n!$, where

$\Delta f(x) = f(x + 1) - f(x)$, and

$\Delta^n f = \Delta(\Delta^{n-1} f)$.

Proof (by induction) that $\Delta^n x^n = n!$ and

$$\Delta^n \left(\sum_0^{n-1} c_i x^i\right) = 0:$$

For n = 1:

$$\Delta^1 x^1 = \Delta x = (x + 1) - x = 1 = 1!$$

and

$$\Delta^1(c_0 x^0) = c_0 - c_0 = 0.$$

Assume true for n = k, i.e. $\Delta^k x^k = k!$ and

$$\Delta^k \left(\sum_0^{k-1} c_i x^i\right) = 0.$$

Then $\Delta^{k+1} x^{k+1} = \Delta^k(\Delta x^{k+1})$

$$= \Delta^k[(x + 1)^{k+1} - x^{k+1}]$$

$$= \Delta^k\left[(k+1)x^k + \sum_0^{k-1} c_i x^i\right]$$

$$= (k+1)\,\Delta^k x^k + 0$$

$$= (k+1) - k!$$

$$= (k+1)!.$$

Also, $\Delta^{k+1}\left(\sum_0^k c_i x^i\right)$

$$= \Delta[\Delta^k(c_k x^k)] + \Delta\left(\Delta^k \sum_0^{k-1} c_i x^i\right)$$

$$= \Delta(k!) + \Delta(0)$$

$$= 0.$$

Thus for all n, $\Delta^n x^n = n!$.

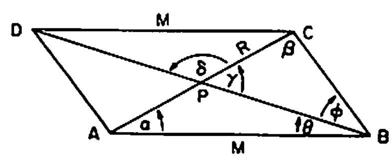
Several respondents noted a connection with higher-order derivatives.

Also solved by Raymond Kinsley, Edward Barton, William Butler, Jr., R. Robinson Rowe, Judith Longyear, Roger Milkman, Gerald Blum, Harry Zaremba, and the proposer, Peter Hadley.

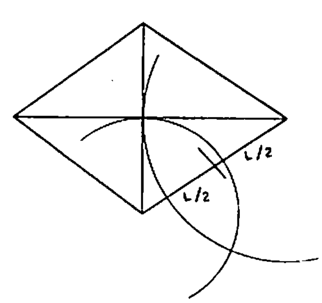
DEC 5 Given two sides, construct a parallelogram whose angles are equal to the angles between its diagonals.

A solution from Raymond Kinsley:

Examination of the parallelogram in question reveals the following:



If $\gamma = \alpha + \beta$, $\delta = \theta + \phi$.
Then, by similar triangles,
Since Δ PCB ≃ ΔCPB with common angle ϕ, then R/S = M/L. Since $\gamma + \phi + \beta = 180°$ and $\gamma + \delta = 180°$, then $\phi + \beta = \delta$. But $\delta = \theta + \phi$; then $\beta = \theta$.
Then, by similar triangles,
Since ΔACB ≃ ΔABP with common angle α, then 2R/L = M/S.
Assume L is the given length. (A second solution appears to exist if it is assumed that the given length is M; however, after a little investigation it is seen that there is no loss of generality in assuming that the given length is L.) Then 2RS/L = RL/S, or $S = L/\sqrt{2}$. For a nontrivial solution to exist, (L − S) < R < (L + S). The construction is as follows:



1. Construct the perpendicular bisector of L.
2. Mark off a distance of L/2 on the perpendicular bisector.
3. Using one end of L as the center, construct an arc with a radius that will pass through the mark made above.
4. Using the other end of L as the center, construct an arc with an arbitrary radius that will intersect the arc made above.
5. Connect two lines from each end of the line L through this point of intersecting arcs.
6. Measure off equal distances on the other side of the point of intersection equal to the length from the end of line L, and the other two points of the parallelogram will have been determined.

Also solved by E. R. Leroy, Irving L. Hopkins, R. Robinson Rowe, William Butler, Jr., Jacob Bergmann, Scott Byron, John Rule, Doug Szper, Harry Zaremba, Winslow Hartford, and the proposer, John Rule.

Better Late Than Never

NS 6 Irving Hale claims to have an alternative solution:

It is true that the ages given in Mr. Blake's solution are correct, but it appears that they are not the only possible combination. They are based on the assumption that the youngest crew members are Peter, Moses, and Abel; but this is not specified in the problem. There are three other possible combinations: that the youngest are Joseph, Moses, and Abel; Joseph, Peter, and Abel; or Joseph, Peter, and Moses. Running the algebra on these combinations results in Peter being −2 years old and 0 years old for combinations 3 and 4, respectively; but for combination 2 (Joseph, Moses, and Abel) the answer is that Peter is 56 and the skipper 83, a result that seems to be consistent with the terms of the problem. My algebra on the two workable age combinations:

Let S = Skipper's age
J = First Mate Joseph's age = 41
P = Navigator Peter's age
M = Deck hand Moses' age = 27
A = Cook Abel's age = 28

$S = 2([J + (P + 14) + M + A]/4)$
$= \frac{1}{2}(J + P + M + A + 14)$
$= \frac{1}{2}(41 + P + 27 + 28 + 14)$
$= \frac{1}{2}P + 55.$

Case I (assuming the three youngest are Peter, Moses, and Abel):

$S + 13 = P + M + A$
$= P + 27 + 28$
$S = P + 55 - 13;$
therefore $\frac{1}{2}P + 55 = P + 42$; $P = 26$; therefore $S = 68$.

Case III (assuming the three youngest are Peter, Moses and Abel):

$S + 13 = J + M + A$
$S = 41 + 27 + 28 - 13 = 83$; therefore $83 = \frac{1}{2}P + 55$, and $P = 56$.

The problem is lovely; certainly the amount of information it generates is awesome. I suspect that the terms of the puzzle could merely be changed to say that the crew voted in the last election and make some sort of subtle reference to "old" Joseph, to eliminate him from the three youngest members.

Additional responses have come as follows:
1977 JUN 2 Stephen Root.
1977 JUN 3 David Ross.
1977 O/N 1 Jerome Gordon and Ruth Turner.
O/N 2 Ed Lynch.
O/N 3 P. Clavier, Ed Lynch, and David Simen.
O/N 4 P. Clavier, Harry Zantopulas, John Rule, Benjamin Rouben, Timothy Maloney, and Leslie Carey.
O/N 5 Neil Hopkins and David Simen.
Y 1977 John Gratwick.

Proposers' Solutions to Speed Problems

M/A SD 1 It is well known that all the routes down a talus pile are irrational. The irrationals form an uncountable infinity.

M/A SD 2 For the evader, $x_e(t) = x_0 + et$. For the pursuer, $x_p(t) = pt$. Capture then occurs at $t^* = x_0/(p - e) = x_{0/p}(1 - \alpha) = t_0/(1 - \alpha)$. The location for capture is $x_p(t^*) - x_0/(1 - \alpha) = x^*$. The condition on α for capture is $\alpha < 1$.

Classified