# Our First Tick Tack Theoretic Problem

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A few issues ago, I was pleased to note the survival of the City University of New York; C.U.N.Y. now appears to have a permanent place in the City. On a much smaller scale, I am pleased to note my permanent place in C.U.N.Y.: I have just received tenure.

I have such a nice letter from William Butler that I cannot resist sharing part of it with you: "... I would like to express my gratitude and indicate that your column does more than fill up space. I have never attended M.I.T. and for that matter have only a high school diploma. I had long ago given up all hope of ever getting a degree or accomplishing anything. A few years ago I started working on 'Puzzle Corner' problems just for the fun of it, and this, in combination with 'pocket' calculators, has rekindled an old aptitude for mathematics. As a result, I have just enrolled in Roger Williams College (as a freshman) and hope to transfer to Brown University in January. Things are going very well academically; the school has made one mistake: they taught me to play with the computer (see the solution to O/N 2 below)."

## Problems

We begin with two problems from previous issues of "Puzzle Corner" that were never solved completely:

NS 10 (formerly 1974 M/A 2) Akbar Ahmed wants a closed form for
$1^1 + 2^2 + \ldots + n^n$.

Since I believe NS 10 is an open problem in number theory, I also include:

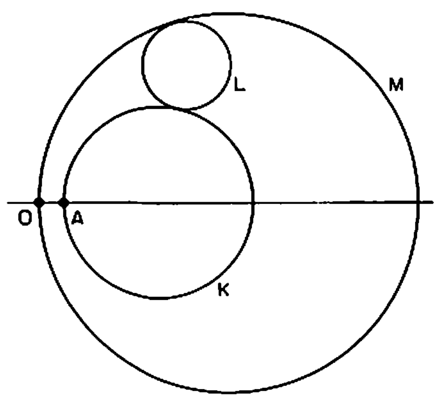
NS 11 (formerly 1974 MAY 3) Hervé Thiriez is interested in square matrices composed of just 0s and 1s. He feels that any such matrix can have determinant no greater than $F_n$ (F for Fibbonacci), where $F_n$ is defined by $F_1 = F_2 = 1$ and for n at least three $F_n = F_{n-1} + F_{n-2}$. There is an example to show that $F_n$ is actually achieved. Is M. Thiriez correct?

FEB 1 Our first new problem this issue is from Steve Grant, who wants you to place one White King, two White Rooks, and one Black King so that White, who is to move, can mate with any of four moves.

FEB 2 Judith Q. Longyear wants to know: When does $[\sqrt{n}]$ divide n where [ ] is the floor or greatest integer function. More generally, when does $[\sqrt[k]{n}]$ divide n?

FEB 3 Sheldon Razin asks the following coloring question: Given an n-by-n checkerboard and $n^2$ checkers of n different colors, and given that there are n checkers of each color, is it possible to arrange all the $n^2$ checkers on the board so that no two checkers of the same color lie in the same row, column, or diagonal? (By diagonal is meant *all* the diagonals, not just the two main diagonals.)

FEB 4 A geometry problem from Harry Zaremba:




In the figure, circle L is tangent to circles K and M, and the shortest distance from its center to the tangent at O is 6.5 inches. If the distance between tangents at O and A is 1 inch, and the radii of circles K and L are 4 and 2 inches, respectively, what is the radius of circle M? What is the locus of the centers of all circles which are tangent to both circles K and M?

FEB 5 Richard J. Alden has sent us our first "Tick Tack" theoretic problem; he writes: I am currently manufacturing and selling a game I invented called "Tick Tack Math." It has 52 playing cards numbered 1 to 40 with 1 to 12 repeated. The mechanics of the game involve using two-card mathematical combinations to equal upturned single cards from the same deck. Addition, subtraction, division, and multiplication are all acceptable combinations. A single card that matches an upturned card can also be played. The question is: Ignoring knowledge of other upturned, played, or held cards, how many practical ways (including permutations) are there of making each number? I say "practical" because a combination like $2 - 1 = 1$, while mathematically correct, wastes a card and should be omitted from the solution.

### Speed Dept.

FEB SD1 The man at the left says, "The man in the middle is Tom." The man in the middle says, "I'm Dick." The man at the right says, "The man in the middle is Harry." Tom always tells the truth. Dick sometimes tells the truth. Harry never tells the truth. Who is who?

FEB SD 2 We close with a problem from John Prussing: At the race track, a bettor commented that one-fourth of the horses finished ahead of the horse he bet on and two-thirds of the horses finished behind. Did his horse finish "in the money"?

## Solutions

The following solutions are to problems published in "Puzzle Corner" for October/November, 1977:

O/N 1 The original problem as printed read, "Given the following hands, show how South can make six spades":

♠Q J 10 9 8<br>♥A 5<br>♦A 10<br>♣A K 9 3

♠A K 4 3 2<br>♥K 4 3<br>♦8 3<br>♣6 5 2

♠7 6 5<br>♥—<br>♦J 9 7 6 5 4 2<br>♣Q J 10

♠—<br>♥Q J 10 9 8 7 6 2<br>♦K Q<br>♣8 7 4

But there was a misprint; the hands are right, but the contract was supposed to be six hearts with the opening lead a high spade. The surprising part is that many readers were able to deduce this on their own; perhaps next issue I'll try one of the problems encoded. The following is from Brian Boyce:

There is no way that South can make

six spades, but here is how he can make six hearts if the opening lead is ♠A: The winning play strips West of everything but his ♠K and a small trump, then squeezes him out of his ♠K on a lead from his partner. At trick one declarer trumps the ♠A with the ♥6; ♥2 is then led and taken in dummy with ♥5 (West will not go up with the ♥K, since to do so makes the contract a breeze for the declarer). Once in dummy, declarer continues to ruff all his spades using the ♦A and the ♣A and ♣K for transportation. The ♦K is cashed and East is put in the lead with a club. At this point East has two diamonds, South has the ♥Q and ♥J (trump), West has the ♥K and ♥4 (trump), and dummy has the ♥A and a small club. East must lead a diamond; South trumps with the ♥J and now West is in a squeeze. If he plays low, South's ♥J holds the trick and the low club is discarded from dummy, leaving ♥A to win the final trick. If West plays his ♥K at trick 12, declarer overtakes with the ♥A in dummy and wins the final trick with the ♥Q in his hand.

Also solved by Elmer Ingraham, Robert Bart, Frank Klotz, Neil Cohen, Richard Hess, Robert Hatton, William Butler, Gordon Bart, Avi Ornstein, Ed Nordstrom, John Rutherford, George Holderness and the proposer, Winslow H. Hartford.

O/N 2 Find three perfect squares such that the sum of any two is a perfect square.

Several readers (including some familiar names) submitted "proofs" that no such numbers exist. However, William Butler sent in the following (non-empty) list of solutions with the three perfect squares less than 1,000,000:

| $\sqrt{A}$ | $\sqrt{B}$ | $\sqrt{C}$ | $\sqrt{A+B}$ | $\sqrt{A+C}$ | $\sqrt{B+C}$ |
|---|---|---|---|---|---|
| 44 | 117 | 240 | 125 | 244 | 267 |
| 85 | 132 | 720 | 157 | 725 | 732 |
| 88 | 234 | 480 | 250 | 488 | 534 |
| 132 | 351 | 720 | 375 | 732 | 801 |
| 140 | 480 | 693 | 500 | 707 | 843 |
| 160 | 231 | 792 | 281 | 808 | 825 |
| 176 | 468 | 960 | 500 | 976 | 1068 |
| 240 | 252 | 275 | 348 | 365 | 373 |
| 480 | 504 | 550 | 696 | 730 | 746 |
| 720 | 756 | 825 | 1044 | 1095 | 1119 |

Many readers noted that if (A,B,C) is a solution so is (NA, NB, NC) for any integral N. Thus there are infinitely many solutions. Harry Zaremba and Frank Rubin have found several of the above solutions by utilizing Pythagorian triples.

Also solved by Bruce Walker, Neil Cohen, Alan LaVergne, Michael Groves, Harvey Goldman, Edward Lynch, Robert Bart and the proposer, Naomi Markovitz.

O/N 3 Let n be a positive integer. A trapezoidal representation of n is a decomposition of n into a sum of consecutive positive integers — e.g., 15 = 15, 15 = 7 + 8, 15 = 4 + 5 + 6, 15 = 1 + 2 + 3 + 4 + 5. How many distinct trapezoidal representations does n have?

The following solution is from Richard Hess:

Pick a number $n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} \cdot \ldots \cdot P_i^{\alpha_i}$, where $P_i$ is its largest prime factor of n. Let $B = 3^{\alpha_2} \cdot \ldots \cdot P_i^{\alpha_i}$; n can be expressed as the sum of k consecutive integers if:

for odd k, k divides n, so k divides;

for even k, n/k is half an odd integer, so k $= 2^{\alpha_1+1}$ r where r divides B.

Thus the number of odd decompositions equals the number of factors of B, namely $(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_i + 1)$. The number of even decompositions is also the number of factors of B. The total number of decompositions is therefore $2(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_i + 1)$. Unfortunately, however, this number includes decompositions which start with nonpositive numbers. Surprisingly, though, exactly half of these decompositions have all positive numbers. Thus, the number of trapezoidal representations of n is $(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_i + 1)$. More precisely, let N be the number of terms in the trapezoidal representation; and let A = n/N be the average value of a term (if N is even, A will be half-integer). In order for there to be all positive terms in the trapezoidal representation, the smallest one must be positive:

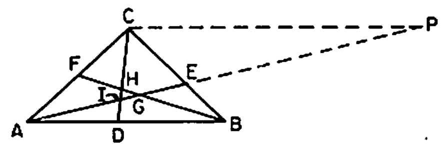
$S = A - (N - 1)/2 > 0 \Rightarrow 2A - N > -1 \Rightarrow 2A - N \geq 0$.

If N is odd we may write the condition as $2n/D - D \geq 0$, where D = N (the odd divisor). If N is even, 2A is an odd divisor (= D), and we can write $D - 2n/D \geq 0$, where D = 2A is an odd divisor. The expression $\pm(2n/D - D)$ is the difference of an even number (2n/D) and an odd number (D) and can therefore never be zero. Thus if we consider each odd divisor of n we generate all the trapezoidal representations, one for each odd divisor as follows:

$2n/D - D > 0 \Rightarrow$ number of terms = D; average term = n/D.

$2n/D - D < 0 \Rightarrow$ number of terms = 2n/D; average term = D/2.

Also solved by William Butler, Edward Lynch, Winslow Hartford, Frank Rubin, Neil Cohen, Judith Longyear, Michael Kennedy, Robert Bart, R. Robinson Rowe, Frank Klotz, Michael Jung, and the proposer, Naomi Markovitz.




O/N 4 Given triangle ABC such that AB/AD = BC/BE = CA/CF = n. Draw AE, BF, and CD intersecting at points G, H, and I. What is the area of triangle GHI?

The following solution is from Frederic Vose:

If AB/AD = BC/BE = CA/CF = n, then triangles ABC and ADC have common altitude bases ∼ n and triangles ADC and ADI have common altitude bases ∼CD/ID. Construct CP parallel to AB, extending AE to P: IC/ED = CP/AD (since triangle AID ∼ triangle CPI), and CE/BE = CP/AB (since triangle ABE ∼ triangle CPE). Then: CD/ID = IC/ID + 1 = CE/BE · AB/AD + 1 = (BC/BE − 1) AB/AD + 1 = n (n − 1) + 1. Triangle AIC has the area of triangle ABC multiplied by $1/n(1 - 1/[n(n-1)+1])$; triangles ABG and CBH have the same area as triangle AIC. Triangle GHI has the area of triangle ABC multiplied by $[1 - 3/n(1 - 1/[n(n-1)+1])] = (n-2)^2/[n(n-1)+1]$ · area ABC.

Emmet Duffy and the proposer, Harold Heins, calculated several additional areas. There were also solutions from Steven Conrad, Harry Zaremba, I. L. Hopkins, Naomi Markovitz, Timothy Maloney, Sam McCluney, Jacob Pomerantz, Neil Hopkins, Neil Cohen, R. Robinson Rowe, William Butler, Richard Hess, and Robert Bart.

O/N 5 If a bridge foursome plays one hand every five minutes, how long will they have to play to have a 1-per-cent chance of a hand repeating? (We require that each person has the same hand that he or she did on any previous deal; but the two deals in question need not be consecutive.)

I particularly enjoyed this problem. Emmet Duffy's response met all the conditions for an ideal solution: accuracy, legibility, ease of typesetting, and use of the same method I chose. Seriously, the first three criteria are the ones I use in addition to favoring unfamiliar respondents. Mr. Duffy's solution follows:

The number of bridge hands is given by 52!/13! · 13! · 13! · 13!, which is equal to $5.3644738 \cdot 10^{28}$. Call this value A. If the probability is 1/100 that a hand will repeat in n hands, then the probability is

99/100 that no hand will repeat in n hands. The probability that the second hand differs from the first is (A − 1)/A. The probability that the third hand differs from the first and second is (A − 2)/A. If the probability is 99/100 that n hands are all different, then:
$[(A-1)/A][(A-2)/A][(A-3)/A]\ldots[A-(n-1)]/A = 99/100$
$(1-1/A)(1-2/A)(1-3/A)\ldots 1-(n-1)/A = 99/100.$
The value A is such a large number that even if (n − 1) is several billion the value (n − 1)/A will be small; hence the approximation, natural log of $(1-x) = -x$, can be used. Taking the natural log:
$-1/A \cdot -2/A \cdot -3/A \ldots -(n-1)/A = \log 99 - \log 100;$
$[1+2+3+\ldots+(n-1)]/A = \log 100 - \log 99 = 0.01005034;$
Summing the arithmetic progression:
$[n(n-1)]/2A = 0.01005034;$
$n(n-1) = 0.02010068 \cdot 5.3644738 \cdot 10^{28} = 10.782957 \cdot 10^{26}.$
The number n will be so large that as an approximation (n − 1) = n. Taking the square root, $n = 3.2837413 \cdot 10^{13}$. Let the year equal 365 + 97/400 days. If a hand is played every five minutes, there will be 288 hands per day and $1.0518984 \cdot 10^5$ per year. Dividing this figure into n results in $3.1217285 \cdot 10^8$ years.

Also solved by Robert Bart, R. Robinson Rowe, Harry Zaremba, Winslow Hartford, Avi Ornstein, Frank Rubin, Michael Kennedy, Timothy Maloney, Neil Cohen, Richard Hess, and the proposer, William J. Butler, Jr.

Better Late Than Never

MAY 1 Jacob Bergmann submitted the following improvement which shows that mate can be forced in six moves; Mr. Bergmann attributes the solution to Henry Dudeney in the 1890s. It's shown in the box above, omitting the sixth checkmating move.

1 P-K4 K any
2 Q-N4

| 2 ... K to KB file<br>3 Q-Q7 | | 2 ... K elsewhere<br>3 Q-N7 | |
|---|---|---|---|
| 3 ... K to first rank | 3 ... K elsewhere | 3 ... K to first rank | 3 ... K elsewhere |
| 4 P-QN3 | 4 P-Q4 | 4 B-N5 | 4 P-Q4 |
| If K on B1: 5 B-N2 | If K on N3: 5 Q-K7 | If K on Q1: 5 Q-Q7 (mate) | If K on R4: 5 Q-QN7 |
| If K on N1: 5 B-B4 (ch) | If K on B3: 5 B-QB4 | If K on B1: 5 B-B6 | If K on N5: 5 Q-QN7 (ch) |
| If K on R1: 5 B-N2 (ch) | If K on R4: 5 Q-KN7 | If K on N1: 5 B-R6 | If K on N3: 5 Q-Q7 |
| | If K on R5: 5 Q-R3 (mate) | If K on R1: 5 B-R6 | If K on B3: 5 B-KB4 |
| | If K on K5: 5 B-Q3 (mate) | | If K on Q3: 5 B-KB4 (ch) |
| | | | If K on K3: 5 B-QN5 |

PERM 2 John Podolsky has responded.
1976 O/N Edward E. Lynch has responded.
Y1977 Scott Byron has responded.
1977 JUN 4 Tom Jenkins, *Technology Review*'s proofreader, notes that the last line of the solution (top of page 16, December) should have been:
16 0 0 0 355/113 ADD
J/A 1 Eric Feldstein and Raymond Ortman have responded.
J/A 2 Nadir Godrej and Robert Bishop have responded.
J/A 5 Nadir Godrej has responded.

Proposers' Solutions to Speed Dept.

FEB SD 1 From left to right: Dick, Harry, and Tom.
FEB SD 2 No; the horse was fourth out of 12.