How Many Buttons on the Calculator?

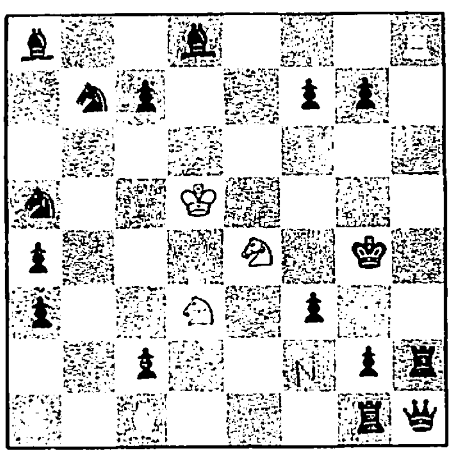
Puzzle Corner
by
Allan J. Gottlieb

As a result of a bombshell received from Frank Rubin, the "Better Late Than Never" section is rather long; so I will limit my introduction to responding to Emmet J. Duffy, who asks if a collection of problems from Puzzle Corner has ever been published. Several of our problems are old classics so have been in print prior to (and after, as well) their appearance in Puzzle Corner. However, no organized collection has appeared. If anyone wonders why, let me add that publishers are not exactly lined up outside my office begging for the opportunity.

Problems

NS-8 We start this month with an old problem that was never completely solved. The challenge, from then-tenth-grader Leslie Servi, first appeared in December, 1970:
Under what additional conditions is it true that $6N + 1$ or $6N - 1$ is prime, where N is a counting number?

J/A-1 Our first regular problem for this month is a chess offering from Bob Kimble. He claims that White can force a draw from the following position, with White to move.



J/A-2 P. V. Heftler wants you to find the smallest number which can be partitioned in six distinct positive integers such that the sum of any five of these six is a perfect square.

J/A-3 Joe Horton asks the following question about calculator design: assume a fixed number of buttons can fit on the calculator. In order to maximize the total number of functions, how many buttons should be "primary" keys and how many should be "shift" keys? (Each "primary" key gives one function if depressed alone and a different one if depressed with different "shift" keys. No more than one "shift" key may be used at once.) After you've answered the question with those rules, try lifting the last restriction.

J/A-4 Greg Jackson asks: How many times must a deck of cards be shuffled before it returns to the same order? Assume 52 cards and a perfect, nonconservative shuffle — one in which the first card on shuffle j is the second on shuffle j + 1.

J/A-5 Karl Kadzielski asks for an integer solution to

$$\frac{A^4 + B^4 + C^4}{A + B + C} = 39.$$

Speed Department

J/A SD-1 Emmet J. Duffy submits the following:
If i is the square root of -1, what is the square root of i?

J/A SD-2 R. E. Crandle has a circle with 256 points in its interior. Can he find a chord that divides the interior into two regions each containing 128 points?

Solutions

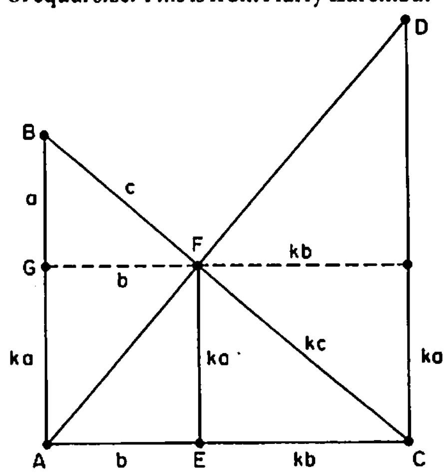
M/A-1 Rodney Yarborough, world's unluckiest bridge player, has been playing bridge for a number of years. During this period he has never received a hand worth even a single point. (Conventional point counting gives four points for each ace, three per king, two per queen, and one per jack. Also, void suits count three, singleton suits two, and doubleton suits one.) Rodney calculates that he has seen more than 1 per cent of the total number of these terrible hands. What is the minimum number of hands that Rodney has seen?

The following counting argument is courtesy of David Simen: Call a bridge hand "bad" if it is not worth any points. Such a hand has at least three cards to each suit, and only 2s through 10s may appear in the hand. All "bad" hands may be constructed as follows: throw out all jacks, queens, kings, and aces from the deck, and separate the remaining 36 cards into the four suits. Next, choose three cards from each suit; there are $\binom{9}{3} = 9!/[3!(9-3)!] = 84$ ways to choose each triple of cards, and so 84^4 ways of choosing the 12 cards. Finally, pick one card from the remaining 24 to complete the hand. There are thus $24 \cdot 84^4$ such constructions. However, we have counted each "bad" hand four times; for, given a "bad" hand, there is no way to distinguish which card of the four-card suit was the last card to be added to the hand. We conclude that the total number of bad hands is $24 \cdot 84^4/4 = 6 \cdot 84^4 = 298{,}722{,}816$; so Rodney has seen at least 2,987,229 of them. I suspect Rodney is exaggerating.

Also solved by Frank Carbin, Emmet J. Duffy, Bob Lutton, Winslow H. Hartford, Steve Grant, William Rosenfeld, Gerald Blum, R. Robinson Rowe, Richard I. Hess, Harry Zaremba, Avi Ornstein, Frank Rubin, Charlie Bahne, and the proposer, William J. Butler, Jr.

M/A-2 Given AB, CD, and EF perpendicular to AC, find any set of integers x, y, m, w, and h such that x = CD, y = AB, m = AD, n = BC, w = AC, and h = ET.

Many solutions are possible to this problem — even one solution with ladders of equal size. This is from Harry Zaremba:



Let the sides of triangle BGF be a, b, and c, and those of triangle FHC be k times as large. From similar triangles AFE and FDH, we have $DH = ka(kb)/b = k^2a$. Also, from triangle ADC, $\overline{AD}^2 = \overline{AC}^2 + \overline{DC}^2 = (b + kb)^2 + (ka + k^2a)^2$, or

$$AD^2 = (1 + k)^2(b^2 + k^2a^2).$$

The required integral distances thus become

$$\begin{aligned}
x &= \overline{CD} = ka + k^2a = ka(1 + k)\\
y &= \overline{AB} = a + ka = a(1 + k)\\
m &= \overline{AD} = (1 + k)(b^2 + k^2a^2)^{\frac{1}{2}}\\
n &= \overline{BC} = c + kc = c(1 + k)\\
w &= \overline{AC} = b + kb = b(1 + k)\\
h &= \overline{EF} = ka.
\end{aligned}$$

My technique to meet the problem requirements is to assume compatible integers for a, b, and c, and then, by trial, select k so that the factor $(b^2 + k^2a^2)$ in the expression for m is a square. If a, b, and c are chosen, respectively, to be 5, 12, and 13, the factor $(b^2 + k^2a^2)$ becomes a square for $k = 7$. Thus one set of integers for the problem conditions is

$$\begin{aligned}
x &= 7 \cdot 5 + 7^2 \cdot 5 = 280\\
y &= 5 + 7 \cdot 5 = 40\\
m &= (1 + 7)(12^2 + 7^2 \cdot 5^2)^{\frac{1}{2}} = 296\\
n &= 13 + 7 \cdot 13 = 104\\
w &= 12 + 7 \cdot 12 = 96\\
h &= 7 \cdot 5 = 35.
\end{aligned}$$

Also solved by Marshall Fritz, William Rosenfield, William J. Butler, Jr., Frank Rubin, R. Robinson Rowe, Richard I. Hess, Gerald Blum, Steve Grant, Winslow H. Hartford, Bob Lutton, Edward Lynch, David Simon, Bruce Fleischer, Everett R. Leroy, Edward S. Talley, Emmet J. Duffy, Jordan S. Wouk, Harry Zantopulos, and the proposers, William F. Cheney and Norman M. Wickstrand.

M/A-3 A rural storekeeper in Georgia has a set of balance scales and a rock weighing 40 pounds. A seller from the city is passing through and he luckily has a set of conventional scales. Seizing upon this opportunity, the storekeeper desires to break his rock up so that he can weigh any exact poundage between one and 40 pounds. The city seller, however, plans to charge outrageous rates for the use of his modern scales. What is the minimum number of pieces into which the storekeeper can break his rock and still accomplish his purpose? How much would each weigh? (Rocks may be placed on either or both trays of the balance scales.)

This problem admits only one solution. As pointed out by Dan Sheingold, this technique has other applications — notably tristate electronics and ternary resistance ladders — see his note in *Analog Dialogue* 9-2 (1975). The solution is 3^0, 3^1, 3^2, 3^3 — i.e., 1, 3, 9, 27.

Also solved by Bruce Fleischer, Winslow H. Hartford, Edward S. Talley, David Simen, Edward Lynch, R. Robinson Rowe, Bob Lutton, Gerald Blum, Richard I. Hess, Frank Rubin, William J. Butler, Jr., William Rosenfeld, Harry Zaremba, Charlie Bahne, David Alan Roe, Emmet J. Duffy, Jim Inglesby, Robert Pogoff, Andrew Egendorf, Leo B. Masters, Jr., Benjamin Rouben, Naomi Markovitz, Bill Swedish, Albert H. Steinbrecher, David Gluss, Mary Lindenberg, George Flynn, Joe Lacey, Raymond Gaillard, and Harry Zantopulos.

M/A-4 Find the fourth term for each of the following related sequences: (a) 1, 20, 190; (b) 1, 21, 210; and (c) 1, 22, 231.

Andrew Egendorf, the entrepreneur behind the first pinball machine in M.I.T.'s Baker House (at which I was the star and undisputed chief crowd-pleaser), submitted the following "solution":

Jack Parsons, the proposer of this problem, will shortly be visited by the C.I.A., since the series is obviously based on the *Washington Post* coverage of the Soviet submarine recovery by Hughes' *Glomar Explorer* early in 1976. In particular, on January ("1") 20th ("20"), on page 19, headline position ("190"), it was reported that a Soviet submarine of unspecified type, code numbered 1910 ("1910"), was scooped up by the Glomar Explorer in 19,090 feet of water. The fourth (a) term is therefore 1910 [and the fifth must be 19090]. The (b) and (c) series are figured by adding the two numbers from the preceeding row which are above and above-to-the-left. The (b) code is used by East Germany, and the (c) code by Albania. The (a) code was ours.

A serious solution came from William R. Rosenfeld:

One infers immediately the three rules:

$$\begin{aligned}
x_{k1} &= 1\\
x_{k2} &= k\\
x_{k+1,l} &= x_{k,l-1} - x_{kl}\ ,
\end{aligned}$$

assuming that $k = 20$ for sequence (a). Not recognizing the recursion law, one uses it (with rule 2) to step back quickly to the neighborhood of $k = 1$. Lo! Recognition dawns! It is our old, familiar friend, Pascal's triangle! Thus $x_{kl} = \binom{k}{l-1}$ and the three terms required are:

$$\binom{20}{3} = 1140,\ \binom{21}{3} = 1330, \text{ and } \binom{22}{3} = 1540.$$

(Serious) solutions were also received from John I. Prussing, Harry Zaremba, William J. Butler, Jr., Richard I. Hess, Bob Lutton, R. Robinson Rowe, Edward Lynch, Bruce Fleischer, Winslow H. Hartford, Naomi Markovitz, Ron Greenstein, Benjamin Rouben, David Gluss, Frank Rubin, Harry Zantopulos, and the proposer, Jack Parsons.

M/A-5 Prove that the sum of the distance from any point in or on an equilateral triangle to the three sides of the triangle is constant.

The following solution was submitted by Jonathan Poritz:

This problem can be divided into three categories. In category one, the point P is in the interior of the triangle ABC and the distances to the sides AB, BC, and AC are a, b, and c, respectively. We know, from the formula ½bh, that the area of triangle ABP is equal to ax/2, where x is the length of a side of triangle ABC. This is because a would be equal in length to the altitude from P to AB. Using this method we can find the areas of triangles ABP, BCP, and ACP, which, when added together, yield the area of triangle ABC. Doing this, we find

$$\begin{aligned}
(ax + bx + cx)/2 &= (x^2\sqrt{3})/4\\
4x(a + b + c) &= 2x^2\sqrt{3}\\
a + b + c &= (x\sqrt{3})/2.
\end{aligned}$$

Since $(x\sqrt{3})/2$ will not change for a different choice of P, so also $a + b + c$ will not change. For categories two and three, in which P is on a side or at a vertex of triangle ABC, the only different in the argument is that one or two of a, b, or c would be zero. This, of course, will still not change $(x\sqrt{3})/2$. Incidentally, this problem was assigned as homework in my ninth-grade geometry class last week.

Also solved by Ken Haruta, Morrie Gasser, Leon Bankoff, Everett R. Leroy, Robert Pogoff, Bill Swedish, Mary Lindenberg, Emmet J. Duffy, Charlie Bahne, Gerald Blum, Steve Grant, Benjamin Rouben, Winslow H. Hartford, Bruce Fleischer, R. Robinson Rowe, Bob Lutton, Richard I. Hess, William J. Butler, Jr., Harry Zaremba, John I. Prussing, William Rosenfeld, Harry Zantopulos, and Frank Rubin.

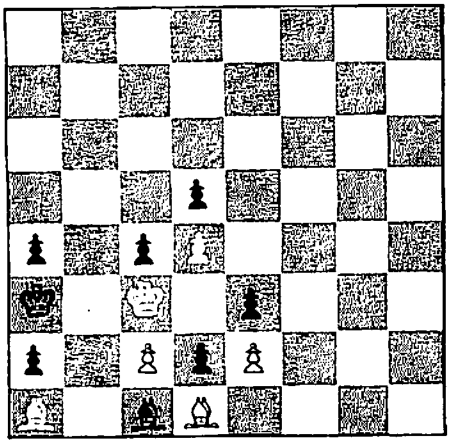
Better Late Than Never

1975 JUN-1 Responses have been received from Gerald Blum and Marc Gottlieb. Let me remind everyone that due to "technical difficulties" (*see "Better Late Than Never" in March/April*) the correct solution is Mr. Butler's published in the "Better Late Than Never" section in February, not the Chandler-Gottlieb solution published earlier in the same February issue.

1976 JUN-5 The following interesting comments are from Gerald Blum:

Your solution has both an error and a flaw. The error is that the equation should have a plus sign, not a minus. The flaw is that it is not a solution to the problem posed. The problem, as has been stated at least three times, is that "He can run twice as fast as *she* can swim." Since she is "struggling," we might resonably assume that she is not swimming — i.e., stationary. However, we know *absolutely nothing* about *his* "speed," or if he can swim at all! (If he can't, he probably should not leave the shore at all, especially if the shoreline is precipitous!) Thus the only reasonable assumption is that he can run (at the same speed) on or through the water; with this assumption, $x = 50$!

J/A-1 Steve Grant has a simplification of Mr. Nelson's remarkable solution as shown in the following diagram:



O/N-1 Gary Schwartz feels that a diamond should be ruffed at trick four; the revised solution (published in "Better Late Than Never" for May) does this.

O/N 4 Howard Ostar notes a typo in his published solution. The term 19y/8 should be 9y/8. Donald Barnhouse has also responded.

O/N-5 The proposer, Dr. Rubin, gives a reference for more information: "The Measure of Recognizable Sets of Real Numbers," *American Mathematical Monthly*, 83, pp. 348-49.

I received several other unconvincing responses. Mark Davidson, however submitted one I believe:

Like many questions concerning the frequency of occurrence of certain substrings of digits in decimal expansions, this is most easily handled using the ergodic theorem. Note that it is enough to show that the set of S-bounded reals in the unit interval has measure 0 since x is S-bounded if and only if x+n is S-bounded, for all integers n. Let I be the open unit interval and J be the set of all one sided sequences, $(w_1\ w_2\ w_3\ w_4 \ldots\ldots)$, where each w_i is a digit from 0 to 9. Let $X_i(x)$ be the ith digit in the decimal expansion of x. (Assume that infinitely repeated 9's are not allowed.) Then we have a map:

$\sigma: I \rightarrow J$

$\sigma(x) = (X_1(x), X_2(x), \ldots\ldots)$.

If $a_1a_2a_3 \ldots a_n$ is a finite string of digits, define the cylinder set $C(a_1a_2a_3 \ldots a_n)$ to be

$\{w \in J : w_1 = a_1, w_2 = a_2, w_3 = a_3, \ldots, w_n = a_n\}$.

There is a natural measure μ on J which is determined by its value on the cylinder sets:

$\mu(C(a_1, \ldots a_n)) = 1/10^n$.

Using this measure on J and Lebesque measure on I, the map σ is an intective measure preserving map. Given a sentential digital form S, it is clear what is meant by saying that a finite string of digits has the form of S: e.g., 43562123 has the for of 4?5~3.

Let $A_S = \cup\{C(a_1 \ldots a_n)$ such that $a_1 \ldots a_n$ has the form of S$\}$. Then $\mu(A_S) > 0$. Let $T : J \rightarrow J$ be the shift to the left:

$T(w_1w_2w_3\ldots) = (w_2w_3w_4w_5\ldots)$.

The point is that S occurs in the decimal expansion of x, starting at the ith digit, if and only if

$T^{(i-1)}[\sigma(x)] \in A_S$.

Now T is ergodic with respect to μ, and the characteristic function of A_S, X, is measurable. The ergodic theorem says that for almost all $w \in J$, as $n \rightarrow \infty$,

$$\frac{1}{n}\sum_{i=0}^{n-1} X[T^i(w)] \rightarrow \int_J X\, d\mu = \mu(A_S) > 0.$$

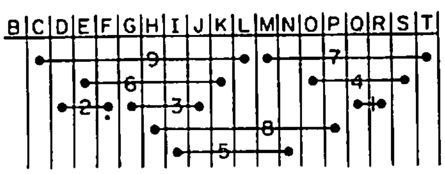
In particular, for almost all w, S occurs in the sequence infinitely often; so for almost all x, S occurs infinitely often in the decimal expansion of x, which shows that the set of S-bounded x has measure 0. Cf. Billingsley, *Ergodic Theory and Information*.

DEC-5 Morton P. Mathew has submitted the following:

The "tennis problem" answered by R. Robinson Rowe in March/April was an actual attempt by some tennis players in Winchester, Mass., to schedule their tennis matches effectively — i.e., with maximum rotation of players. I worked out "seat-of-the-pants" solutions for up to 24 players but never found a neat, general method of attack. Nonetheless, 20 regulars are playing tennis in Winchester with the following schedule: Players A through T start with match I in 5 doubles courts:

I.	$\frac{AB}{IN}$	$\frac{CL}{DF}$	$\frac{EK}{OS}$	$\frac{GJ}{HP}$	$\frac{MT}{QR}$
II.	$\frac{AC}{JO}$	$\frac{DM}{EG}$	$\frac{FL}{PT}$	etc.	
III.	$\frac{AD}{KP}$	$\frac{EN}{FH}$	etc.		

In match II, every position but A is replaced by the succeeding letter player (B by C, I by J, ... T by B). Similarly in match III. After 19 matches, every player has had every other player for a partner once and for an opponent twice. It was laboriously worked out in this way: Obviously, the alphabetic spacing of all partners must be different at the start. If D and F, and also G and I, were partners in match I, then by match IV D and F positions would be replaced respectively by E and G, then F and H, and then G and I, who were already partners in match I. To avoid this, a lot of trial and error can produce this chart with line-connected dots under letters indicating pairing of partners, all varying distances apart on the alphabetic scale (B has A for his initial partner).



So far so good. The rotation can be completed without partners repeating. It is considerably harder to complete the arrangement so that the alphabetic spacing of opponents does not repeat more than once. For example, if the starting line-up included:

I. $\frac{CL}{HP}$ $\frac{DF}{IN}$ $\frac{EK}{GJ}$

rotation to match II would give:

II. $\frac{DM}{IQ}$ $\frac{EG}{JO}$ $\frac{FL}{HK}$

and to match III:

III. $\frac{EN}{JR}$ $\frac{FH}{KP}$ $\frac{GM}{IL}$

and already E has had J for an opponent three times. This was inevitable from the start, where C opposite H, D opposite I, and E opposite J were all five alphabetical steps apart. Systematic shuffling can eventually bring in the result. It can also consume untold hours. A computer program could undoubtedly handle it. But the original problem, to find a straight path to a general solution for 4n players, seems to remain.

I received a phone call from another reader suggesting that a general solution will be submitted within a month. Maybe this won't be an NS problem in the 1980s after all!

1977 JAN-2 This is starting to sound like a broken record: "The proposer, Frank Rubin, refutes the published solution." Recall that the published solution had a leading zero. Many readers (and the Editor) found this unacceptable but the best we could do; I was disappointed that a Rubin problem should have an unaesthetic solution. Now Dr. Rubin replies:

I do not know whether you have changed the wording of this problem, but it would have been clearer to say, "Replace each letter by a unique digit from 0 to 9 to make a correct equation." Of course, many readers will note that there is no solution in base 10. Moving right along, there are two solutions in base 11:

940A98 × 7 = 5A66A21

and

923496 × A = 8411475

but both use the digit A = 10. In base 12 there are five solutions:

A923A5 × 7 = 634430B
562158 × 9 = 4177130
412549 × A = 35005B6
40A847 × B = 3899825
64096A × B = 5988932

Of these, all but one use digits beyond 9. The only valid solution in base 12 is then

```
  562158
×      9
 4177130.
```

Winthrop M. Leeds, Richard I. Hess, Kenneth L. Wise, and Gerald Blum have responded.

Proposers' Solutions to Speed Problems

J/A SD-1 In polar coordinates, i is 1 at an angle of 90°. The square root is 1 at an angle of 45°, which in rectangular coordinates is cos 45° + i (sin 45°) = $\sqrt{2}/2 + (i\sqrt{2})/2$.

J/A SD-2 (courtesy of the Editor) There are at most $\binom{256}{2}$ lines that contain two (or more) of the 256 points. Choose a line L, not parallel to any of these. Start L to the left of the circle and move it to the right parallel to itself. By construction, L crosses points one at a time. Stop after crossing 128 points.

Allan J. Gottlieb studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973); he is now Assistant Professor of Mathematics and Coordinator of Computer Activities at York College of C.U.N.Y. Send problems, solutions, and comments to him at the Department of Mathematics, York College, Jamaica, N.Y. 11451.