Flipping "Super-Cents" for the Old Howard

Puzzle Corner
by
Allan J. Gottlieb

Space was short, and "Better Late Than Never" was too long for the March/April issue; most of it was postponed until this issue. My only defense: better late than never!

JAN 1 was improperly stated; it is revised in this month's "Solutions" section.

I'm asked to announce that the fifth annual Mathematics and Statistics Conference of Miami University, Ohio, will be held there (Oxford, Ohio, 45059) on September 30 and October 1, 1977; the theme is "Number Theory — Pure and Simple"; Friday papers will be directed at college teachers and researchers, and those on Saturday are for high school teachers and students at all levels. For further information write Stanley E. Payne at Miami's Mathematics Department.

Problems

We continue the practice of presenting previous problems never completely solved.

NS7 This problem, from Frank Rubin, appeared as 1970 Oct/Nov 2. R. Robinson Rowe sent in a (very) partial solution and some months later Judith Q. Longyear and Michael Rolle responded, but I do not believe a complete solution ever appeared.

Let N be some fixed positive integer. Show that there exist positive rational numbers $a_1, \ldots, a_N$ such that for any m, $1 \le m \le N$

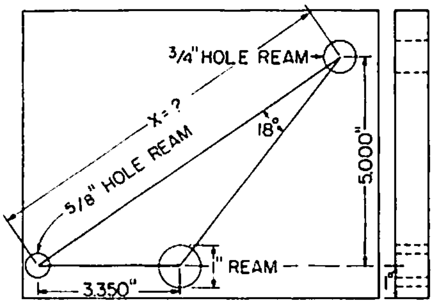
$$S(m) = \sum_{i=1}^{m} a_i^3$$

is the square of a rational number, and $S(N) = 1$.

MAY 1 A chess problem from Gary A. Ford:

White begins a chess game with the usual set-up, while Black has only his king (in the usual position). What is the minimum number of moves White needs to achieve mate, assuming Black tries to avoid it? Is the first man moved by White the same in all minimal solutions?

MAY 2 Neal MacClaren used to be Director of Quality Control at Brown and Sharpe Manufacturing Co., a large machine tool firm. He enclosed a typical drill pattern problem (below); solve for x.



MAY 3 R. Robinson Rowe recalls that "a whimsey" at M.I.T. 60 years ago was, "Let's flip a penny: heads we go to The Nip, tails to the Old Howard, and if it lands on edge we stay here and study." He speculates that The Nip may be gone after 60 years, recalling that Cambridge was then a local-option dry town and The Nip was a respectable saloon in Boston, much patronized by students, located on Tremont Street between Park Station and Boylston Street. The "Old Howard" was a burlesque theater, a bit less respectable than its only competitor, Waldron's Casino, located between Scollay Square and Haymarket Square. At the Old Howard, he says, the chorus line wore tights a/c Boston "blue laws," and The Nip served draft beer for five cents a schooner and mixed drinks two-for-a-quarter. Mr. Rowe's question:

Suppose we had flipped a "super-cent" made of N cents epoxied together in a cylindrical form. Given that the diameter of a cent is 12 times its thickness, what is the nearest integer N for (a) The "super-cent" with equal changes of *heads, tails,* or *edge*? (b) The "super-cent" with equal chances of *face* or *edge*?

MAY 4 An interesting problem from Paul Mailman concerns palindromes (numbers like 181 or 247742, i.e. $a_0 a_1 \ldots a_n$ with $a_i = a_{n-i}$ $0 \le i \le n$). Mr. Mailman writes:

Have you ever seen a proof that if you reverse a number and add it to itself, and repeat that process, you will eventually get a palindrome? I've heard that it's true, and have found no counter-examples, but can't prove it.

359	79
953	97
1312	176
2131	671
3443	847
	748
	1595
	5951
	7546
	6457
	14003
	30041
	44044

MAY 5 Russ Nahigian will undoubtedly get me into trouble with this since I've played backgammon even less often than either bridge or chess — but here goes. He writes:

I offer the following problem to those who know the rules of backgammon. Given the original board setup, play any 3 initial dice of your choosing such as 6-6, 5-5, 6-1, etc. to lock up the 2 black pieces in the corner of your home table with a perfect prime. That is, the black pieces in the corner are locked in such that they cannot move even 1 space (or point) with the 6 openings in front blocked by closed points. (Assume the black corner pieces stay in one place during the 3 rolls of the dice.)

(Anyone in the Boston area interested in joining a backgammon club should call Mr. Nahigian at 617-494-2015 days or 617-648-6219 evenings.)

Speed Department

MAY SD1 The following, entitled "Fun at the Factory," is from Smith D. Turner: Punch any number (try your weight) into a calculator in the Rad Mode, and then punch the cos key repeatedly; alternate punches give two series that converge to give

0.7390851332

Try another number — say your phone number — and you get the same! Thus, a constant emerges that, like e, depends only on the process and not the specific data. What is this new fundamental constant of nature I have discovered?

MAY SD2 Gregory James Ruffa wants all the positive x's such that X^{X^X} gives the same answer whether evaluated top down or bottom up.

Solutions

JAN 1 As noted above, this problem was incorrectly stated. South is to make *four* of the five tricks against any defense. The correct problem reads:

With a no-trump contract, how can South, who is on lead, make four of the remaining five tricks of the following hand against any defense?

	♠ 8 ♥ Q 7 ♦ 9 ♣ 10	
♠ A ♥ 10 9 6 ♠ 8 ♣ —		♠ — ♥ — ♦ A 7 ♣ Q 9 6
	♠ — ♥ J ♦ J 5 ♣ A 8	

Try again, folks.

JAN 2 Replace each letter by a unique decimal digit to make a correct equation:

```
 ROBERT
×     F
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KENNEDY
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The answer is

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 192516
×     3
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0577548
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George H. Ropes and Bill Swedish each submitted this solution without saying how it was obtained, R. Robinson Rowe came close, and the "Allan J. Gottlieb Arithmetic Endurance Award" has been sent to Dennis Sandow to be given to his SR-52 which discovered the solution after *218 hours* of calculating. They would have gone after a uniqueness proof but that would have taken 29 days (and then the solution would have been "better late than never").

JAN 3 Prove that if $X, \ldots, X_n$ are distinct real numbers (distinct natural numbers are all right, too) and $n > 1$, then the finite sequence $\{Xi\}_{i=1}^{n}$ has a monotone subsequence of length greater than $\sqrt{n}$.

This difficult problem has been nicely solved by Ron Moore:

In the problem statement we must change "greater than" to "greater than or equal to," since if $n = m^2$ is a perfect square, a sequence of length n may be constructed whose longest monotone subsequence is of length m. The sequence consists of m blocks of length m:

$$m, m-1, \ldots, 1\ ,\ 2m, 2m-1, \ldots,\ m+1, \ldots, m^2, m^2-1, \ldots,\ m^2-m+1$$

Any decreasing subsequence must be entirely contained in some block and so has at most m elements; any increasing subsequence can contain at most one element from each block and thus has at most m elements. Given the distinct numbers $X_1, \ldots, X_n$, we list them in this order from left to right and form a directed graph G_0 by drawing an arc from X_i to X_j if $i < j$ and $X_i < X_j$:

$$X_1\ X_2 \ldots X_{n-1}\ X_n$$

If there is an arc from X_i to X_j, call X_i a predecessor of X_j; also, the number of arcs terminating at a vertex will be called its input valence. Let S_0 denote the set of points with input valence 0 in graph G_0. (Note that S_0 is not empty since certainly $X_1 \in S_0$.) An element of S_0 is less than all elements to its left; so if $|S_0| \geq \sqrt{n}$, the elements of S_0 form a decreasing subsequence of length $\geq \sqrt{n}$. If $|S_0| < \sqrt{n}$, form a new graph G_1 by removing from G_0 all elements of S_0 and all arcs emanating from them. Let S_1, denote the set of vertices with input valence 0 in graph G_1. Again, S_1 is non-empty, and each element of S_1 is less than all points to its left in G_1. So, if $|S_1| \geq \sqrt{n}$, the elements of S_1 form a decreasing subsequence of length $\geq \sqrt{n}$; if $|S_1| < \sqrt{n}$, define graph G_2 by removing from G_1 all elements of S_1 and all arcs emanating from them. This process continues until either

(a) some set S_i is found with $|S_i| > \sqrt{n}$, in which case the elements of S_i form a decreasing subsequence of length $\geq \sqrt{n}$; or

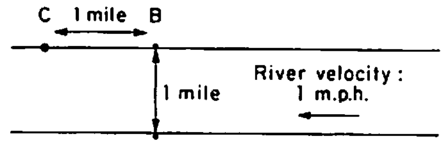
(b) the original graph G_0 is exhausted.

If case (a) never occurs, then, as $S_0, S_1, \ldots$ are successively removed, less than $\sqrt{n}$ points are removed at each step. After $m = [\sqrt{n}]$ steps, less than $m\sqrt{n} \leq n$ points have been removed from the original graph G_0, so G_m is non-empty and the set S_m is non-empty.

Now observe that, given an arbitrary element P of S_i $(0 < i \leq m)$, there is a predecessor p^1 of P where $P^1 \in S_{i-1}$. (For by construction P has input valence 0 in graph G_i but positive input valence in G_{i-1}; so there is an arc $P^1 \rightarrow P$ where P^1 is in G_{i-1} but not G_i, i.e., $P^1 \in S_{i-1}$.) Consequently, we can choose an element Y_m of S_m arbitrarily and successively choose $Y_{m-1}, \ldots, Y_0$ such that $Y_i \in S_i$ and Y_i is a predecessor of Y_{i+1}. Then $Y_0, Y_1, \ldots, Y_m$ is an increasing subsequence of length $m + 1 = [\sqrt{n}] + 1 > \sqrt{n}$.

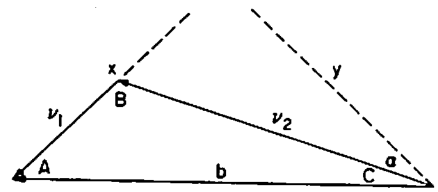
R. Robinson Rowe and Roger Milkman have also responded.

JAN 4 A swimmer, who swims at a constant rate of two miles per hour relative to the water, wants to swim directly from point A to a point C, which is one mile downstream and on the other side of a river one mile wide and flowing one mile per hour. At what angle should he point himself, relative to the line AB (perpendicular to the river)?



The paucity of solutions to the last two problems is counterbalanced by the outpouring of responses to this one. The following is from Eugene Bedal:

First obtaining a general formula seemed to help avoid mixing of values. Per diagram,



let $V_1 = 1$, velocity of river
$V_2 = 2$, velocity of athlete
x, y = 1 each, the coordinates given.

Then $b = \sqrt{x^2 + y^2}$

and $\mathrm{Sin}\ A = y/b = y/\sqrt{x^2 + y^2}$.

From the Law of Sines:

$(\mathrm{Sin}\ C)/V_1 = (\mathrm{Sin}\ A)/V_2$, and

$\mathrm{Sin}\ C = (V_1\ \mathrm{Sin}\ A)/V_2$
$= V_1/V_2 \cdot y / \sqrt{x^2 + y^2} = 1/2\sqrt{2}$

Thus $\alpha = 45° - \mathrm{Csc}^{-1}\, 2\sqrt{2}$.

Also solved by 31 other readers (the list is simply too long to print), including the proposer, Ted Mita, and an anonymous employee of General Mills Chemicals.

JAN 5 An ounce of gold can be drawn into 50 miles of wire or hammered into a sheet of 100 square feet. Which is thicker, the wire or the sheet? Hint: Gold, unlike currencies, does not float.

This problem also received a large response; I selected the following from Carl M. King:

According to my old *Handbook of Chemistry and Physics*, the density of gold is 19.26 gm/cc. The thickness of the gold sheet I calculate to be:
$t = 6.8316 \times 10^{-6}$.
The diameter of the gold wire I calculate to be:
$d = 1.9884 \times 10^{-4}$,
which means that the wire diameter is the greater by a factor in excess of 29. I note that this factor is a function of the density of the gold. For a lesser density the factor would be less, so I wondered what would be the density of a material for which the wire diameter would be equal to the sheet thickness. Or to state the question in another way: Is there a value of density where the answer to this problem re-

verses, and the thickness of the sheet would be the greater? So I set the ratio to unity, and let the density be the unknown. To satisfy this condition I find:
density = .023 gm/cc
which is an impossible condition at normal room temperature and pressure. It is impossible, since the lightest element known is hydrogen gas (H_2), and it has a density of .090 gm/cc. So the answer in general terms is that the diameter is greater than the thickness, as those terms are defined in the problem, regardless of the density of the gold.

Also solved by 21 readers (once more the list is too long to print), including the proposer, Homer D. Schaaf, and the General Mills Chemicals employee.

Better Late Than Never

1973 MAY 3 A fine analysis has come from R. C. Lacher:

In essence, the problem was to decide when the symbol

(*)

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$$

makes sense. The solution by R. R. Rowe (and evidently also Bob Baird, Walter Hill, Neil Judell, Peter Kramer, Albert Mullin, Harry Nelson, John Prussing, and Harry Zaremba) which appeared in the October/November 1973 issue, is not complete. By concentrating too intensely on the upper end of the convergence interval, these fellows evidently missed perhaps the most interesting property of "Infinitely Stacked Exponents": the symbol (*) is *not* convergent for very small values of x.

To facilitate further explanation, I will make some definitions:

Let $E_1(x) = x$ and $E_{n+1}(x) = x^{E_n(x)}$ for $n = 1, 2, \ldots$ and $x > 0$; and definite

$\ell(x) = \lim_n \inf E_n(x)$

$u(x) = \lim_n \sup E_n(x)$

for $x > o$. Rowe *et al.* discovered that $l(x) = +\infty$ for $x > \sqrt[e]{e}$ and $u(x) < +\infty$ for $0 < x \le \sqrt[e]{e}$. Here are some other facts:

$$\ell(x) = \lim_k E_{2k+1}(x) \text{ and} \qquad (1)$$

$$u(x) = \lim_k E_{2k}(x) \text{ for } 0 < x \le \sqrt[e]{e}.$$

Both $\ell(x)$ and $u(x)$ are continuous on the interval

$$0 < x \le \sqrt[e]{e}. \qquad (2)$$

$$\ell(x) = u(x) \qquad (3)$$

if and only if $\frac{1}{e^e} \le x \le \sqrt[e]{e}$.

Thus, the symbol (*) is defined if and only if $\frac{1}{e^e} \le x \le \sqrt[e]{e}$. The "solution" to the equation

(**)

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = a$$

by replacing (**) with the equation $x^a = a$ is valid as long as the symbol (*) makes sense, i.e., as long as

$$\frac{1}{e} \le a \le e.$$

The red herring in this problem is that the equation $x^a = a$ makes sense for *any* $a > 0$! J. J. Andrews and I discovered the above facts in connection with our solution to the simultaneous functional equations

$$x^{a(x)} = b(x), \; x^{b(x)} = a(x)$$

(paper to appear in *Aequationes Mathematicae*).

1975 JUN 5 In the *February* issue solutions to this appeared under *both* "Solutions" and "Better Late Than Never." Although this sounds impossible, it's true. When the *March* issue was submitted, the BLTN was dropped due to space limitations; but meanwhile room "appeared" in February, so in it went although for that issue it was not late. At any rate, the author of that (correct) response was William J. Butler, Jr. This obsoletes the (incorrect) Gottlieb-Chandler version given in the solutions section (and I thought topology was complicated). Responses have subsequently been received from Kevin Czuhai, Draper Kauffman, John F. Chandler, and Philip E. Oshel.

1976 JUN 1 Gary Schwartz from the Charles S. Draper Laboratory informs me that when *D-Notes* ran this problem credit was given to *Test Your Bridge Play* by Edwin B. Kantar.

Albert J. Fischer and Charles E. Blair claim to have better ways to play the hand. Mr. Fisher's follows:

The solution proposed by Mr. Ingraham is almost identical with my own. However, there is a hidden trap. Consider the following distribution:

	♠ 4 2 ♥ K 10 8 7 6 ♦ A J 5 ♣ Q 4 3	
♠ 8 7 5 ♥ 2 ♦ Q 9 8 3 2 ♣ J 9 7 6		♠ K J 9 6 ♥ 4 3 ♦ K 10 7 6 ♣ K 10 5
	♠ A Q 10 3 ♥ A Q J 9 5 ♦ 4 ♣ A 8 2	

Mr. Ingraham's solution consists of drawing two rounds of trump ending in his hand and then leading low to the ♣Q. He states that if East can win the ♣Q with the ♣K, he is still all right so long as either:
(a) East has the ♠K and ♠J, or
(b) West has no more than the ♠J and two small spades. Let us examine each of these propositions:
(a) East has the distribution shown above, which includes ♣K, ♠K, and ♠J. Assume for convenience that East returns a small club after he has won his ♣K (any minor suit card will do). South wins with the ♣A and finds himself in an almost impossible situation:

	♠ 4 2 ♥ K 10 8 ♦ J 5 ♣ 4	
♠ 8 7 5 ♥ — ♦ Q 9 8 ♣ J 9		♠ K J 9 6 ♥ — ♦ K 7 6 ♣ 10
	♠ A Q 10 3 ♥ Q J 9 ♦ — ♣ 8	

He can pick up three tricks in the spade suit by means of the double finesse, but he needs the trump suit as the entry each time, and he will no longer have enough trump left over to come to 12 tricks: he has four tricks in the bank which, when added to two more trump tricks, three more spade tricks, and two ruffs (one in each hand), will total only 11. Mr. Ingraham's error was in counting a second diamond ruff, which cannot be taken since the trump is needed as transportation to dummy for the double finesse in spades.

In the position shown, the only chance I can see for South is the forlorn hope that West has been dealt the singleton ♠K. South should cash the ♠A, dropping the offside King (and causing West to move several feet back from the table), cross to dummy with a trump, and finesse against East's ♠J. The losing club goes on the ♠Q, and South trumps two black cards in dummy and two red ones in his hand for a total of 12 tricks: four in the bank, three spaces, one trump (transportation), and four ruffs.

(b) Everyone has the distribution shown originally except that East's ♠J has been exchanged for one of West's small ones. Mr. Ingraham states that he intends to arrive at 12 tricks by means of "three high spades, one spade ruff, two diamond ruffs, the ♣A, and two more trump tricks." I do not see how South can come to "three high spades" with this initial position — a finesse of the ♠Q followed by ♠A and a spade ruff will drop the ♠J but not the ♠K. It appears to me that this line of play suffers from the same problems as before — namely, an insufficient supply of tricks.

The simple solution to the transportation dilemma is to draw only one round of trumps, ending in hand, prior to leading the low club toward the Queen. Now South truly is in command of the situation. He wins the club (or diamond) return in hand, crosses to dummy with a trump, and takes the double finesse in spades. When this wins, he can return to dummy with a third round of trumps to repeat the finesse. At this point he still has two trumps in each hand and can cross-ruff for the remainder of the tricks.

1976 JUN 2 Stephen F. Wilder noticed that Sally may be speaking nonsense and when I suggested this problem for the York biweekly puzzle, my colleague Joe Malkevitch immediately asked "what shape is the table." But then he's a geometer.

1976 JUN 5 B. Rouben and William J. Butler have responded.

1976 O/N 1 R. Robinson Rowe has a method with a slightly higher probability of success:
Of major importance is the distribution of the four trumps held by defenders. The measure of probable distribution is the set of binomial coefficients:

One hand	4	3	2	1	0	
Other hand	0	1	2	3	4	
Probability	1	4	6	4	1	in 16ths.

Those four cards are Q, 10, 9, and 8, and it is important to consider the support of the Q. The possibilities and probabilities may be tabulated:

One hand	Other hand	Chance	Success factor	Success product
Q x x x		.125	0	0
Q x x	x	.375	0	0
Q x	x x	.375	1	0.375
Q	x x x	.125	0.547	0.068
		1.000		0.443

"Chance" is derived from the distribution table above. Note that 3,1 and 1,3 each had 4/16 chance, adding to ½; since any one of the four cards might be the singleton, this ½ is divided to ⅜ for Qxx and ⅛ for Q alone. "Success Factor" is the probability of success for each distribution. For Qxxx and Qxx, the factor is 0, because the Q will take one trick and another will be lost in a minor suit. For Qx, trumps are drawn without loss, two clubs are ruffed in dummy after giving up a club, and the contract is airtight— that is, the success factor is 1. Finally, if the Q is alone it falls, but the J in dummy must be used to pick up the last outstanding trump. Now declarer's best chance is to set up a diamond 13-er. Dummy holds AKXXX and he can ruff twice. Opponents hold seven diamonds; if divided 4-3, they will fall. The chance of this is determined by the binomial coefficients:

One hand	7	6	5	4	3	2	1	0	
Other hand	0	1	2	3	4	5	6	7	
Probability	1	7	21	35	35	21	7	1	in 128ths

Thus the chance of a 4-3 or 3-4 distribution is 70/128 = 0.547, which is the success factor and the success product is 0.068 as shown. This is additive to the total success probability of 0.443. More precisely, this is 227/512, which equals 0.443 359 375. For this strategem to work, one must make sure of entry cards in dummy and closed hand. For either contingency, Qx or Q, the north-south play to the first 6 tricks would be, with an underscore marking a led card and asterisk marking a winning card:

Trick	North	South
1	♥ 2	♥ 5
2	♥ 6	♥ 8
3	♦ A*	♦ 2
4	♦ 6	♠ 4*
5	♠ 2	♠ A*
6	♠ 3	♠ K*
Then if trumps fell,		
7	♣ 3	♣ A*
8	♣ J	♣ 4,
losing, and for club or heart return		
9	♠ 7*	♣ 5
10	♦ K*	♣ 6
11	♦ 7	♠ 5*
12	♠ J*	♣ 10
13	♦ 8	♠ 6*
But if singleton Q fell,		
7	♠ J*	♠ 5
8	♦ K*	♣ 4
9	♦ 7	♠ 6*
10	♣ 3	♣ A*
11	♣ J	♣ 5
losing, and on any return		
12	♠ 7*	♣ 6
13	♦ 8*	♣ 10

winning 70/128 of the time.

In the first alternative, at Trick 9, I showed the play for a club or heart return. Opponents have no spades. If a diamond is returned, ♦K wins, as for Trick 10, and dummy has an unneeded trump to play later.

1976 O/N 2,3 William J. Butler, Jr., has responded to both.

1976 O/N 4 William J. Butler, Jr., Naomi Markovitz, Boyan Baldwin, and B. Rouben have responded.

1976 O/N SD2 Roger Stern feels that the proposed method would be difficult to carry out.

1976 DEC 2 Naomi Markovitz, George H. Ropes, Judith Q. Longyear, Robert F. Barnes, Scott W. Peterson, David J. Pogoff, K. Haruta, and James Larson have responded.

1976 DEC 3 James Larson, K. Haruta, Judith Q. Longyear, Winslow H. Hartford, Naomi Markovitz; Hiroshi Ono, William E. Cooper, Gene Bedal, and Harvey Elentuck have responded.

1976 DEC 4 Judith Q. Longyear, Naomi Markovitz, Winslow H. Hartford, K. Haruta, and James Larson have responded.

1976 DEC 5 Winslow H. Hartford and Judith Q. Longyear have responded.

Y1976 Bill Swedish notes that $43 = 1^9 + 7 \cdot 6$.

1977 FEB SD1 William Hornick has taken this problem seriously. The problem asked which common (non-archaic) English verb had no infinitive, and the answer was given as "can" in the sense of "to be able." Mr. Hornick offers eight other common, non-archaic English verbs which also have no infinitive forms and explains why:

1. Could	as in	I could go . . .
2. Would	as in	I would go . . .
3. Should	as in	I should go . . .
4. Will	as in	I shall go . . (He will go . . .)
5. May	as in	I may go . . .
6. Might	as in	I might go . . .
7. Ought	as in	I ought to go . . .
8. Used	as in	I used to go . . . (when it means "formerly")

All eight of these (plus can), are auxillary and/or operator verbs. They are never used without another verb (or verb form). Of course, most times the other verb form is implied rather than expressed. (Ex. "Are you going to write a letter?" " I should.") Implied with "I should" is "I should write." Furthermore, the first six in my list are always dependent on conditional (in a grammatical sense) situations. The last two need not be.

PERM 2 Smith D. Turner ($\int dt$) prefers a variant where *any* mathematical function except greatest integer is allowed and the object is to use as few 4s as possible. He can make 0 to 9 using a total of 13 4s.

Proposers' Solutions to Speed Problems
Both solutions are courtesy of the Editor:
SD1 The unique solution of $\cos(x) = x$.
SD2 For $x > 0$,

$$X^{(x^X)} = (X^X)^X \text{ implies } X^{(x^X)} = X^{X \cdot X}$$

implies $X = 1$ or $X^X = X^2$
implies $X = 1$ or $X = 2$.

Allan J. Gottlieb studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973); he is now Assistant Professor of Mathematics and Coordinator of Computer Activities at York College of C.U.N.Y. Send problems, solutions, and comments to him at the Department of Mathematics, York College, Jamaica, N.Y. 11451.

Classified