How to Celebrate Election Day and a New Year, Too

Puzzle Corner
by
Allan J. Gottlieb

How fitting to be writing a puzzle column on Election Day 1976 — a real puzzle!

Last month I had a pleasant surprise. One of my wife's colleagues, Scott Brodie, came to tell us that Martin Gardner had again mentioned "Puzzle Corner" in his highly respected column in *Scientific American*. I should like to thank him for the generous comment.

A final remark before getting down to serious business: the backlog of "speed" problems is still low.

Problems

Y1977 Since this is the January issue, we once again present a "yearly" problem — this time Y1977. You are to take the digits 1, 9, 7, and 7; and the operators +, −, * (multiply), / (divide), and ** (exponentiation); and form the integers from 1 to 100 using each digit once (7 is thus used twice) and the fewest possible number of operators. Parentheses may be used to indicate the order of operations and, in case of a tie in the number of operators, a solution using 1 9 7 7 in order is to be favored.

The answers to Y1976 appear under "Solutions" below. The deadline for Y1977 is November 1 so that we may report the solutions next January.

JAN 1 We begin with a five-card bridge problem from Emmet J. Duffy: With a no-trump contract, how can South, who is on lead, make a remaining five tricks against any defense?

	♠ 8 ♥ Q 7 ♦ 9 ♣ 10	
♠ A ♥ 10 9 6 ♦ 8 ♣ –		♠ – ♥ – ♦ A 7 ♣ Q 9 6
	♠ – ♥ J ♦ J 5 ♣ A 8	

JAN 2 To accord with our political theme, I offer the following criptarithmetic puzzle from Rank Rubin: Replace each letter by a unique decimal digit to make a correct equation:

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 ROBERT
×     F
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KENNEDY
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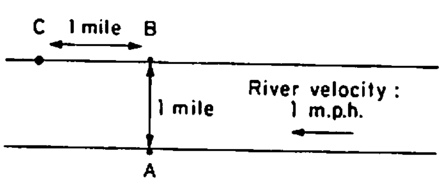
JAN 3 A number-theory problem from R. E. Crandall: Prove that if X, . . ., Xn are distinct real numbers (distinct natural numbers are all right, too) and $n > 1$, then the finite sequence $\{X_i\}_{i=1}^{n}$ has a monotone subsequence of length greater than $\sqrt{n}$.

JAN 4 An athletic problem from Ted Mita: A swimmer, who swims at a constant rate of two miles per hour relative to the water, wants to swim directly from point A to a point C, which is one mile downstream and on the other side of a river one mile wide and flowing one mile per hour. At what angle should he point himself, relative to the line AB (perpendicular to the river)?



(Getting the equation right is a sufficient solution, since one needs tables to obtain the final answer.)

JAN 5 To again honor the elections, we close with a problem based on international monetary exchange submitted by Homer D. Schaaf: An ounce of gold can be drawn into 50 miles of wire or hammered into a sheet of 100 square feet. Which is thicker, the wire or the sheet? Hint: Gold, unlike currencies, does not float.

Speed Department

JAN SD1 People say Jimmy Carter is very calculating. Speaking of calculators, Don Savage submits the following: One consequence of the recent purchase of my first calculator was loss of sleep from staying up to the wee hours merrily pushing buttons. So here is a three-part quickie for my fellow calculator freaks:

I. $\frac{1}{9 \times 9} = ?, \frac{1}{9 \times 99} = ?, \frac{1}{9 \times 999} = ?, \frac{1}{99 \times 99} = ?, \frac{1}{999 \times 999} = ?, \frac{1}{9999 \times 9999} = ?$

II. $\sqrt{e} \overset{?}{=} \sqrt{\frac{100}{11}}$

III. $\frac{104348}{33215} = ?$

JAN SD2 Norman Spencer has the spirit: On the first day of Christmas my true love gave to me a partridge in a pear tree. On the twelfth day she gave me 78 gifts. How many gifts did I receive altogether?

Solutions

Y1976 Take the digits 1, 9, 7, and 6; the operators +, −, * (multiply), / (divide), and ** (exponentiate); and form the integers from 1 to 100 using each digit once and the lowest possible number of operators. Use parentheses to indicate the order of operation, and in case of a tie a solution using 1 9 7 6 in order is favored.

Several people arrived at the following solution:

Number	*Score*	
1	1*	1^{976}
2	1	71 − 69
3	3*	1/[(9-7).6]
4	1	76/19
5	2	$6 - 1^{97}$
6	2*	$(1^{97}) \cdot 6$
7	2*	$(1^{97}) + 6$
8	2	$(1^{96}) + 7$
9	2	$(1^{76}) \cdot 9$
10	2	$(1^{76}) + 9$
11	3*	[(1 + 9) + 7] − 6
12	2	96/(1 + 7)
13	2	(79 − 1)/6
14	2	(17 − 9) + 6
15	1	91 − 76
16	2	(97 − 1)/6
17	2	71 − (9 · 6)
18	1	79 − 61
19	2*	19 · (7 − 6)
20	2*	(19 + 7) − 6
21	3	[(1 · 9) − 6] · 7
22	3*	[(1 · 9) + 7] + 6
23	2	(7 · 6) − 19
24	1	91 − 67
25	1	96 − 71
26	3	[7 · (6 − 1)] − 9
27	3	[(7 − 1) · 6] − 9
28	1	196/7
29		No solution
30		No solution
31		No solution
32	2*	(19 + 7) + 6
33	3	[(1 · 7) · 6] − 9
34	3*	(1 − 9) + (7 · 6)
35	3	$7 \cdot (6 - 1^{9})$
36	1	97 − 61
37	2	(9 · 6) − 17
38	3	[9 · (6 − 1)] − 7
39	3	[(1 + 7) · 6] − 9

40	3	$[(1 + 6) \cdot 7] - 9$
41	3	$[(9 - 1) \cdot 6] - 7$
42	3*	$1^9 \cdot 7 \cdot 6$
43	3	$(7 \cdot 6) + 1^9$
44	3	$9 - [(1 - 6) \cdot 7]$
45	2	$(61 - 9) - 7$
46	3	$[(9 \cdot 6) - 1] - 7$
47	2	$(9 \cdot 7) - 16$
48	1	$67 - 19$
49	2	$(16 - 9) \cdot 7$
50	3	$[(9 - 1) \cdot 7] - 6$
51	2	$17 \cdot (9 - 6)$
52	1	$69 - 17$
53	3	$[(1 + 9) \cdot 6] - 7$
54	3	$(1^7 \cdot 9) \cdot 6$
55	3	$[(9 - 1) . 6] + 7$
56	2	$(71 - 9) - 6$
57	1	$76 - 19$
58	2	$(1 . 67) - 9$
59	2	$(1 - 9) + 67$
60	3	$(1^7 + 9) \cdot 6$
61	2*	$19 + (7 \cdot 6)$
62	2	$(1 \cdot 69) - 7$
63	1	$79 - 16$
64	3*	$[(1 + 9) \cdot 7] - 6$
65	3*	$1 + (9 - 7)^6$
66	2	$(76 - 1) - 9$
67	2	$1^9 \cdot 67$
68	2*	$(1 - 9) + 76$
69	2	$1^7 \cdot 69$
70	2	$1^7 + 69$
71	2	$17 + (9 \cdot 6)$
72	2*	$(19 - 7) \cdot 6$
73	2	$(1 \cdot 79) - 6$
74	2	$(1 + 79) - 6$
75	2	$(9 - 1) + 67$
76	2*	$1^9 \cdot 76$
77	2*	$1^9 + 76$
78	2	$(91 - 7) - 6$
79	1	$96 - 17$
80	2	$1^6 + 79$
81	1	$97 - 16$
82		No solution
83		No solution
84	2	$(9 - 1) + 76$
85	2*	$(1 \cdot 9) + 76$
86	1	$19 + 67$
87		No solution
88	2	$(96 - 1) - 7$
89	2	$(1 \cdot 96) - 7$
90	2	$(1 + 96) - 7$
91	2*	$(1 \cdot 97) - 6$
92	2*	$(1 + 97) - 6$
93	2	$(17 \cdot 6) - 9$
94		No solution
95	1*	$19 + 76$
96	2	$1^7 \cdot 96$
97	2	$1^7 + 96$
98	2	$1^6 + 97$
99	2	$(17 - 6) \cdot 9$
100		No solution

Asterisks mark the favored solutions in which 1 9 7 6 appear in order.

Responses were received from Gerald Blum, Albert S. Knight, W. Robert Dresser, Abraham Schwartz, Glenn Rowsam, A. W. Collins, Avi Ornstein, E. Jamin, Robert K. Kennedy, Harry Zaremba, Richard Rudell, R. Bart, A. Holt, Edward Pierce, and William J. Butler, Jr.

J/A 1 Since a clarification was given in the last issue, the solutions will appear in March/April.

J/A 2 It is well known that Pythagorean triples with integers a, b, h, where $a^2 + b^2 = h^2$, may be generated by taking $h = p^2 + q^2$, $a = p^2 - q^2$, and $b = 2\,pq$, where p and q are any integers such that $p > q \geq 1$. A special series of such triples is that in which the legs a and b differ by one unit. By means of a table of squares we can find that the first four of such triples are:

p	q	h	a	b
2	1	5	3	4
5	2	29	21	20
12	5	169	119	120
29	12	985	697	696

(For the tenth such triple, p = 5741, q = 2378, h = 38,613,965, a = 27,304,197, and b = 27,304,196.) If we take the fractions q/p from the table above, they are 1/2, 2/5, 5/12, and 12/29. These are the continued fraction approximants for $\sqrt{2} - 1$. Prove that all the approximants (a) result in Pythagorean triples (b) with legs differing by unity, and (c) that the dimensions increase nearly geometrically with a ratio approaching $(3 + 2\sqrt{2})$.

I received several accurate responses to this problem, and I selected R. Robinson Rowe's due to its clarity and simplicity:

As a continued fraction,

$$\sqrt{2} - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

indefinitely, and if at any stage the part in the oval is the preceding convergent m/n, this succeeding convergent is

$$C = \frac{1}{2 + m/n} = \frac{n}{m + 2n}$$

If that preceding convergent generated a triple $a^2 + b^2 = h^2$ with $a - b = \pm 1$ by the parameters $m = q$, $n = p$, $a = p^2 - q^2$, $b = 2pq$, $h = p^2 + q^2$, we have

$$p^2 - q^2 - 2pq = \pm 1 = n^2 - m^2 - 2mn$$

Then in the succeeding convergent,

$$\begin{aligned} q &= n, \\ p &= m + 2n, \\ a &= (m + 2n)^2 - n^2 = 3n^2 + 4mn + m^2 \\ b &= 2n(m + 2n) \qquad = 4n^2 + 2mn \\ a - b &= -n^2 + m^2 + 2mn = -(\pm 1) \end{aligned}$$

That is, the triple generated by the succeeding convergent will also have legs differing by unity, but a and b will alternate as the larger. This proves tasks (a) and (b) — since any two values for p and q generate triples. We note now that, as above, if one convergent is m/n, the next is n/(m + 2n), that is, each numerator is the preceding denominator. As the ratio of these convergents approaches a limit r, such that n = mr, we have for consecutive convergents

$$\frac{m}{mr} \text{ and } \frac{mr}{mr^2}$$

and by the equation above for C,

$$\begin{aligned} mr^2 &= m + 2mr \\ r^2 - 2r - 1 &= 0 \\ r &= 1 + \sqrt{2}. \end{aligned}$$

Then, from $b = 2pq$, consecutive b's and $2m^2r$ and $2m^2r^3$ with a ratio of

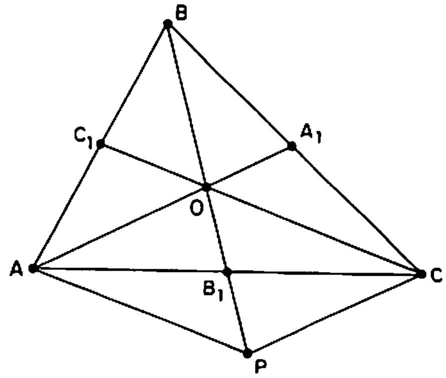
$$r^2 = (1 + \sqrt{2})^2 = 3 + 2\sqrt{2}. \qquad \text{qed}$$

Also solved by Charles J. Rife, John Bonomo, Winslow H Hartford, Gerald Blum, John F. Chandler, and the proposer, Irving L. Hopkins

J/A3 Construct a triangle given the lengths of the three meridians.

Harry Zaremba had little trouble with this one:

Let the given means be M_A, M_B, and M_C, each of which is trisected, initially. Armed with the foreknowledge that the intersection of the means is a point of their trisection and is the centroid of the triangle to be constructed, we proceed as follows (see figure):



(1) Draw line BB, whose length equals M_B.
(2) With B_1 as the center and $M_B/3$ as the radius, locate points O and P on line BB_1.
(3) With points O and P as centers, and $2M_A/3$ and $2M_C/3$ as the radii, respectively, locate point A.
(4) Similarly, using O and P as centers, and $2M_C/3$ and $2M_A/3$ as the respective radii, locate point C.
(5) Draw the figure AOCP, and line AC. Since AP = OC, and AO = PC, the quadrangle is a parallelogram. Diagonals AC and OP bisect each other at point B_1, and line AC is one side of the required triangle.
(6) Draw lines AB and BC. The figure ABC is the triangle required. Lines AA_1 and CC_1, extensions of AO and CO, respectively, are equal to the means M_A and M_C.

Also solved by John F. Chandler, R. Robinson Rowe, William J. Butler, Jr., Mary Lindenberg, Ernest W. Thiele, and the proposer, Robert Pogoff.

J/A4 Write down in a row the coefficients of the expansion of $(a + b)^N$ where N is any integer. Multiply each coefficient by the second coefficient and set the result below the multiplied coefficient. Continue by multiplying by the third, fourth, and fifth coefficients, etc. An example is shown for the coefficients of the expansion of $(a + b)^4$:

1	4	6	4	1
4	16	24	16	4
6	24	36	24	6
4	16	24	16	4
1	4	6	4	1

Find a formula for the sum of the digits in any diagonal.

Neil Cohen's clever method utilizing expansions of $(a + b)^S$ makes this look almost trivial:

Let A_{ij}^N be the coefficient in the ith row and jth column of the array created by the coefficients of the expansion $(a + b)^N$. Then $A_{ij}^N = C(N, i - 1)\,C(N, j - 1)$. By symmetry, it is sufficient to consider diagonals beginning with a coefficient on row 1 and slanting to the left. Pick the coefficient in the kth column. Then S, the sum =

$$\sum_{l=0}^{k-1} A^N_{(l+1,k-l)} = \sum_{p=0}^{k-1} C(N,l)C(N,k-l-1).$$

However, $(a + b)^{2N} = (a + b)^N (a + b)^N$. Therefore, $c(2N,k) =$

$$\sum_{l=0}^{k} C(N,l)C(n,k-l)$$

since the lefthand side is the coefficient of $a^k b^{(2N-k)}$ in the expansion of $(a + b)^{2N}$ and the right hand side is the coefficient of $a^k b^{(2N-k)}$ obtained by expanding $(a + b)^N$ and multiplying the resulting polynomial by itself. Therfore, $S = C(2N, k - 1)$.

Also solved by John F. Chandler, Willaim J. Butler, Jr., R. Robinson Rowe, Harry Zaremba, Gerald Blum, Winslow H. Hartford, William G. Hutchinson, Jr., Frank Carbin, Joseph G. Haubrich, Naomi Markovitz, and the proposer, Emmet J. Duffy.

J/A5 Given a decanter of wine but no measuring devices, find a way to divide the wine among any number of people so that each is satisfied that his/her share is (at least) a fair one. Assume that each person is willing to divide the wine and accept any of the portions.

An excellent response was received from Hal R. Varian, who says there are at least two solutions:

(1) One person pours the wine until someone yells "stop." The person who yells gets the serving. (Of course the pourer can also yell "stop.")

(2) Person one pours a glass of wine and passes it to person two. Person two can either pass it on untouched or pour some back into the decanter. So it goes on down the line. When the last person has had his chance to do this, the glass is awarded to the person who poured last. (Of course this could be the person who originally poured the glass.) The person who gets the glass is out and the process is repeated until only two people are left; then one divides and the other chooses. Mr. Varian notes that "several eminent mathematicians have worked on this problem, among them Banach and Steinhaus. Some of their work is described in Spanier and Dubins' 'How to Cut a Cake Fairly,' *American Mathematical Monthly*, Vol. 68, 1961, and Kuhn, H., 'Games of Fair Division,' in Shubik, ed. *Essays in Mathematical Economics*, Princeton University Press, 1967. I became interested in this subject while in graduate school at Berkeley and ended up writing my thesis on some formal models of distributive justice. A student of mine at M.I.T., Vincent Crawford, considered a variation of this problem in his 1976 Ph.D. thesis. Suppose that the item to be divided is nonhomogeneous — for example, a cake with icing. The two people involved in the division process may have different tastes for icing and cake, so they will evaluate different slices differently. It is not hard to show that divide and choose still has the "no envy" property — neither person will prefer the other person's piece to his own. The question Crawford asked is: In such a game is it better to be the divider or the chooser? Assume that each person knows the other person's tastes completely."

Also solved by Neil Cohen, Joseph G. Haubrich, R. Robinson Rowe, William J. Butler, Jr., Stephen Polloch, Jacob Bergmann, Neil E. Hopkins, Harris Hyman, and the proposer, Stuart Schulman.

Better Late Than Never

1975 DEC 2 Winthrop M. Leeds has responded with the following problem:

In the July/August issue you reported that David R. Kramer has solutions for K = 5 and K = 13 and asked for a procedure for all K. For large values of K where a strictly analytical solution does not seem practical, I suggest using the following procedure:

Draw a large circle, centrally located in the unit square, which intersects the sides of the square at a distance "x" from a corner where "x" is about 0.25 or 0.30. Completely cover one exposed corner of the square with N small circles with the pattern chosen for a minimum amount of overlap with each other and the edges of the corner space. Since all corners are assumed to be covered in the same way, then K = 4N + 1. Now measure the radius of each small circle and express it as a decimal fraction of "x," or kx. The radius of the large circle is $r = \sqrt{(.5)^2 + (.5 - x)^2}$. Now the sum of the areas of all K circles (call it A) can be expressed as a function of "x." Then calculate dA/dx, equate it to zero and solve for "x." If this is not too much different from the originally estimated value, then the value of A for this value of "x" will be very close to the minimum value of the total circle area for the chosen pattern. Here are some of my answers:

K = 13	A = 1.188
K = 21	A = 1.146
K = 37	A = 1.133
K = 45	A = 1.123

What would you think of asking your readers to find the smallest value for K that will yield a minimum for A = 1.100 or less?

[Any takers? — Ed.] You have asked what the procedure should be for infinitely large values of K where A should approach 1. I suggest that you start with an inscribed circle of radius 1, then put a tangential circle in each corner, continue with tangential circles in each uncovered crevice, and as the crevices and added circles get smaller and smaller:

$$\begin{cases} K \rightarrow \infty \\ A \rightarrow 1. \end{cases}$$

FEB 4 Bruce Andeen "strongly disagrees" with the solution published in October/November:

The error is in assuming that the displacement of the ship has any bearing on the water required to operate a lock. The amount of water that must be supplied to (or drained from) a lock to raise (or lower) a given ship is solely equivalent to the volume of the lock. The ship's displacement has already been accounted for, since when the ship entered the lock, the water in the lock is in equilibrium with the body of water from which the ship sailed. Only changes in the ship's displacement occuring *during* the lock operation affect the water requirement. Thus, the correct answer to the problem is *not* $2V_L - V_s$, as stated, but rather $2V_L$.

M/A Mazie Porter has responded.

MAY 3 Lowell Kolb has responded.

J/A SD 1 Neil Hopkins points out the interesting possibility:

$$2^2 + (3/2)^2 = (5/2)^2;$$

A, B, and C are all "limited to primes."

O/N SD 1 Several readers point out that the position could not occur in a game (even after you put the White King somewhere). What could the position have been before White mated? The only defense is that the position was not required to be "game born". While game born positions do exist they A: require the White King, and B: are not nearly so pretty.

Solutions to Speed Department

SD1 I won't ruin your fun by spilling the beans. Buy a calculator and find out for yourself, or "test one out" at Sears.

SD 2 1 + (1 + 2) + (1 + 2 + 3) + . . . + (1 + . . . + 12) = 364

The proposor claims that finite difference techniques yield a general solution

$$\frac{n \cdot (n^2 + 3n + 2)}{6}$$

But that sort of stuff is forbidden in the Speed Department.

Allan J. Gottlieb studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973); he is now Coordinator of Computer Activities at York College of C.U.N.Y. Send problems, solutions, and comments to him at the Department of Mathematics, York College, Jamaica, N.Y. 11451.