Horopters and the Price of Cheese

Puzzle Corner
by
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Hello again. Autumn has just begun and the leaves are starting to turn. This is my favorite season and also the time when I miss New England the most. As the long-time followers of this column know, I spent a year in Santa Cruz, California (on the coast between San Francisco and Big Sur). Although the year-long climate there is the best I have ever experienced, my autumn vote still goes to New England.

This is also the beginning of a new school year. I remain with the (smaller) City University of New York. Perhaps I erred last year in listing my address as Coordinator of Computing. While that is an accurate title, I am still a faculty member in the Mathematics Department and have not been "promoted out of teaching" as several readers have been led to believe.

Now let me congratulate two active contributors to Puzzle Corner. Professor John E. Prussing has been named to a half-time revolving assistant deanship in the College of Engineering of the University of Illinois at Urbana-Champaign. Harry Nelson is now the editor of the *Journal of Recreational Mathematics.* Anyone wishing details should write to him at 4259 Emory Way, Livermore, Calif., 94550.

Two comments on Speed Problems: first, I could use more; second, the proposer's solution (if any) is given in the same issue as the problem (at the end of the column). Other solutions received to speed problems are mentioned only in exceptional cases.

I feel that J/A 1 was well stated. Many readers, however, misunderstood the problem. Let me clarify the point of confusion. You may *not* assume logical play. So for the sub-problem A = 2, B = 3, for example, you must find a position satisfying: 1) If White moves first, *any* sequence of legal moves leads to a White victory (this is A = 2); and 2) If Black moves first, *any* sequence of legal moves leads to a black victory (this is B = 3). Similar remarks hold for the other eight cases. Solutions will be given in the March/April issue along with those to this month's regular problems.

NS 4 is obviously nontrivial, as all answers received so far are wrong. Everyone has "proved" that all three are homeomorphic; this is false.

I received a copy of the *Alphabetic Number Table* from one of its authors, DB. In case anyone is interested, of the numbers from 0 to 1000, three is the 788th in alphabetical order, whereas III is the 903rd in the list of alphabetical roman numbers from 1 to 1000.

Unsolved Problem

Our first problem is another from the past. This never-solved problem first appeared in March, 1969:

NS 5 If a pair of triangles is not co-polar, the joins of corresponding vertices form a triangle and so do the intersections of corresponding sides. The original pair of triangles has been transformed into a second pair which can be transformed into a third and so on. How does the sequence of pairs of triangles behave?

New Problems

DEC 1 Our first new problem is from Frank Rubin, who writes: "Here is a new chess problem, which boasts an incontrovertible solution (indeed, two such solutions) and thus may help you keep out of trouble";

How should eight queens be placed on the chess board so that the total number of moves is maximum?

Mr. Rubin writes that this problem "has an obvious solution, which is eight non-attacking queens, a very well-known position. Needless to say, since this is a Rubin problem, the obvious solution is obviously wrong."

DEC 2 We continue with an offering from Winslow H. Hartford: An interesting series is the "paired" series: (1,1), (2,3), (5,7), (12,17), (29,41), (70,99), . . .

a. What is the next pair?

b. What is the "rule" for constructing the series?

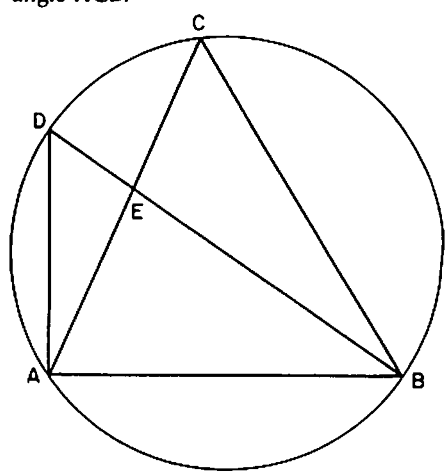
c. Show that the pairs give solutions to the equation:

$$2n_1^2 \pm 1 = n_2^2$$

and that the plus and minus signs alternate. The limit is, of course, $\sqrt{2}$ for the ratio of the pairs.

DEC 3 A geometry problem from Joe Horton:

Stated without proof in a psychology text was an interesting theorem about the horopter. The horopter is the set of points in space which, in traveling from one to another, subtend equal retinal angles. The horopter in a horizontal plane is a circle which passes through the centers of the lenses. In order to show this the necessary theorem was: Given a circle with a chord drawn, the vertical angle of any triangle constructed on that chord will be equal. Phrased that way, of course, it is not true — but if the constraint that the triangles be on the same side of the chord is invoked, then it is true. The problem: prove that given the circle shown, angle ADB = angle ACB.



DEC 4 Dave Kaufman poses an important problem in computer science (called the binary Gray Code):

For any positive integer n, there are 2^n distinct binary numbers of n binary digits (bits), leading zeros allowed. Here is the list for n = 2, arranged in counting order:

00
01
10
11

In decimal, they would read 0, 1, 2 and 3. In order to solve some problems in digital design, the list should be arranged so only *one* bit changes from line to line, including the wrap-around case (bottom line back up to top line).

Here's one solution for n = 2:

00
01
11
10

Problem: find a method for generating such a sequence for arbitrary n.

DEC 5 We close with a tennis problem from Norman M. Wickstrand:

There are 4n tennis players who wish to play 4n-1 doubles matches, where n equals any positive integer. How can the matches be arranged so that all players play in every match with the limitation that each player plays with each other player once only and against each other player the same number of times? When n equals one the solution is easy and quite obvious. Is there a general solution or formula or system? Is it limited to perhaps n equals 5 or 6?

Speed Dept.

SD 1 Our first speed problem is from R. Robinson Rowe:

As an exercise preparing for adoption of the metric system, interpret the following nine quantities as units to become obsolete; their initials, in order, will spell a non-metric unit which will survive:

20.12 cm.
2.54 cm.
118.3 $cm.^3$
10.16 cm.
907180 g.
91.44 cm.
45.72 cm.
40468726 $cm.^2$
502.92 cm.

SD 2 We close with a quickie from Sam Gutmann: Joe Gourmet and Jane Gourmand made a bet (about a question of fact). The loser was to pay the winner one pound of Brie. Yet the stakes (not the odds) were two to one. Impossible?

Solutions

JUN 1 The dealer is North, and East-West are vulnerable:

North:
♠ 4 2
♥ K 10 8 7 6
♦ A J 5
♣ Q 4 2
South:
♠ A Q 10 3
♥ A Q J 9 5
♦ 4
♣ A 8 3

The bidding:

North:	South:
Pass	1 ♥
3 ♥	6 ♥ (!)
Pass	

The opening lead is ♦ 3; trumps are divided 2-1. Plan the play.

The answers I received look different and I'm far from an expert at bridge (as you doubtless know). Thus I am choosing Elmer Ingraham's primarily due to its clarity:

In a bridge game with this problem I would see that 12 tricks are sure if both black Kings are under their Queens, chancy if both are in the same unseen hand, impossible if both are over the Queens.

I would plan the play to discover if there is a sure way to go and to exploit whatever chances may develop along the way, much as detailed below:

1. I will play ♦A, 4, ♥6, 5, and ♥7, A to win the first three tricks, then lead ♣3 toward ♣Q.

2. If East can play ♣K to win the fourth, no return will hurt me if he has ♠K and ♠J in his hand or if West has no more than ♠J, x, x, for I will win all the rest in timely order with three high spades, one spade ruff, two diamond ruffs, the ♣A, and two more trumps for 12 tricks.

3. (a) If West has ♣K he may choose to hold off the fourth trick; ♣Q will win and I will lead ♠2 thru East at the fifth; if West can win with ♠K I can win any return and in timely order all the rest if ♠J will appear in three demands, for 12 tricks made.

(b) If West has held off the fourth and cannot win the fifth I will cross to Dummy to lead ♠4 thru East at the seventh to win ♠A and ruff ♠3 at the eighth; if both ♠K and ♠J have then appeared I will win the rest in timely order for 13 tricks made, or if ♠10 cannot hold I win 12 tricks.

4. (a) If West plays ♣K to win the fourth trick he can return any spade I can claim, or win, all the rest in timely order with two high spades, two high clubs, two spade ruffs, two diamond ruffs and one more trump for 12 tricks made and the contract.

(b) If West has won the fourth and chooses to return any club or diamond, I can win all the rest in timely order if East has ♠K, for 12 tricks bid and made.

Responses were also received from R. Robinson Rowe, William J. Butler, Jr., Neil Cohen, and Richard I. Hess.

JUN 2 Four people — Kevin, Deb, Breck, and Sally — occupied a side table at a recent banquet. During the height of the festivities one of them slipped behind a nearby drapery and a few minutes later emerged to streak down the center aisle and across in front of the head table, out into the corridor, and back to the drapery — shortly to return and reoccupy the vacated chair. When the excitement had died down somewhat the chairman of the banquet committee questioned the four to find out who had reoccupied the vacant chair.

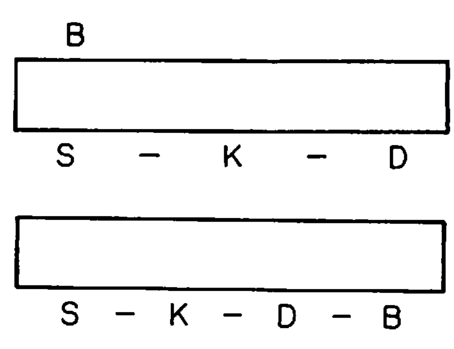
Kevin answered:
1. I sat next to Deb. It wasn't her.
2. Breck or Deb sat to my right.
Deb said:
3. I sat next to Breck.
4. Kevin or Breck was on Sally's right.
Breck replied:
5. I sat across from Sal.
Sally said:
6. Only one of us is lying, and he or she is the guilty one.

Who was the varmint without a garment?

Michael Bissell had little trouble locating the streaker:

Assume Sally is lying. Sally's statement consists of two parts: A: "One of us is lying"; and B: "He or she is the guilty one." Sally would be lying if either part "A" or "B" of her statement is false or if both are false. A false part "A" implies that more than one is lying. A false "B" implies that he or she is innocent. The combinations of a false part "A" and a true or false part "B" can be eliminated since they result in a nonsensical statement. The remaining false statement which Sally could make would be "One of us is lying, and he or she is innocent." This combination requires that Breck, Kevin, and Deb are telling the truth, which is impossible since Breck's statement "I sat across from Sal," Deb's statement "Kevin or Breck was on Sally's right," and Kevin's statement "I sat next to Deb," can result in only the one seating arrangement shown below if all statements are true. However, this arrangement precludes Deb's statement "I sat next to Breck" from being true. Therefore, we can conclude that Sally is not lying.

Since Sally is not lying, there is only one liar and he or she is the guilty one. Assume Kevin is lying. However, since he made the statement "It wasn't her" in reference to Deb, it is implied that Deb is guilty, which contradicts Sally's statement. Therefore Kevin's is not lying. Assume Deb is lying. But this contradicts Kevin's statement that it wasn't Deb. Therefore, the only remaining alternative is that Breck is lying. The only non-contradictory seating arrangement assuming Breck is the liar is shown below. Sally's statement implies that Breck is the varmint without a garment.



Also solved by Mary Lindenberg, Harry Zaremba, Charles Rozier, Alan Baumgardner, Neil Cohen, Richard I. Hess, William J. Butler, Jr., Gregory James Ruffa, R. Robinson Rowe, R. Bart, and the proposer, James Cassidy.

JUN 3 Given a 4 x 4 array with markers in all but one of the squares (as shown); the object is to remove all markers but one by jumping horizontally or vertically (no diagonal jumps allowed), or else to prove that it cannot be done.

Perhaps this problem should have explicitly defined the word jump. However, I feel that the checkers (or Chinese checkers) terminology should apply. Thus you cannot "jump off the board." With this restriction the task is impossible, as Jeffrey Kenton cleverly shows:

Color the squares in the array as indicated:

1 Red	2 Blue	3 Green	4 Red
5 Green	6 Red	7 Blue	8 Green
9 Blue	10 Green	11 Red	12 Blue
13 Red	14 Blue	15 Green	16 Red

Notice that for any legal jump the starting square, middle square, and end square are all different colors. This coloring scheme is essentially unique. Let R, G, B represent the number of occupied red, green, and blue squares, respectively, in the initial position. Similarly, let R′, G′, B′ represent the final position. Let r, g, and b represent the number of moves (jumps) which *end* on red, green, or blue squares.

Then, $R' = R + r - g - b$

$G' = G + g - r - b$

$B' = B + b - r - g$

or, $\Delta R \equiv R - R' = g + b - r$

$\Delta G \equiv G - G = r + b - g$

$\Delta B \equiv B - b' = r + g - b$

which gives $2r = \Delta G + \Delta B$

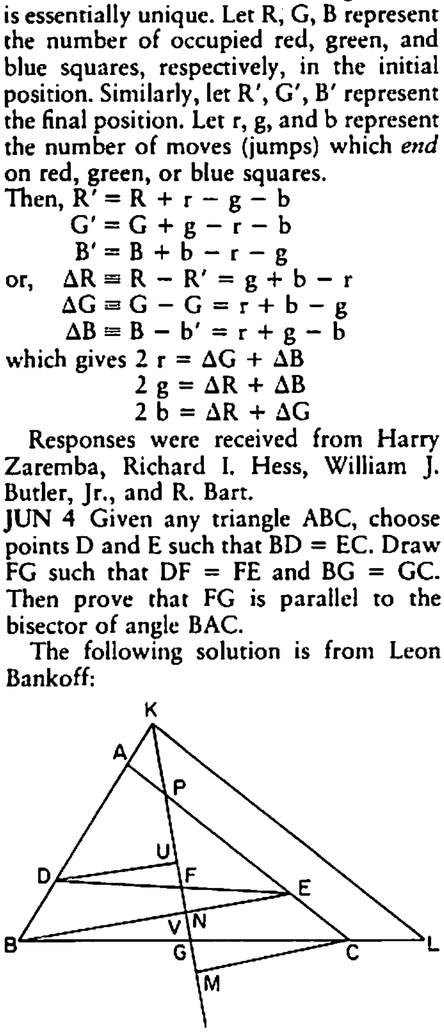
$2g = \Delta R + \Delta B$

$2b = \Delta R + \Delta G$

Responses were received from Harry Zaremba, Richard I. Hess, William J. Butler, Jr., and R. Bart.

JUN 4 Given any triangle ABC, choose points D and E such that BD = EC. Draw FG such that DF = FE and BG = GC. Then prove that FG is parallel to the bisector of angle BAC.

The following solution is from Leon Bankoff:



Extend FG to cut AE in P and the extension of BA in K. Draw KL parallel to AC, cutting BC produced in L. Since triangles ABC and KBL are similar and similarly placed, the proof is now reduced to showing that KG bisects angle BKL. Let α and β denote angles BKG and GKL (or GPE), respectively. Drop the perpendiculars CM, EN, BV, and DU upon KG. Then it is seen that CM = BV and EN = DU, which is equivalent to

PC $\sin\beta$ = BK $\sin\alpha$ (*) and

PE $\sin\beta$ = DK $\sin\alpha$. It follows that

$$\frac{PC}{BK} = \frac{PE}{DK} = \frac{PC - PE}{BK - DK} = \frac{EC}{BD} = 1, \text{ or}$$

PC = BK, which, substituted in (*), yields $\sin\beta = \sin\alpha$.

Hence KG bisects angle BKL, and FG is parallel to the bisector of angle BAC.

Also solved by R. Bart, Richard I. Hess, William J. Butler, Jr., Harry Zaremba, Ronald Goldman, John C. Gray, Burnham H. Dodge, R. Robinson Rowe, Neil Cohen, John Purbrick, and the proposer, J. Harvey Goldman.

JUN 5 Recall that this problem was modified in the October/November issue; the solution will therefore appear in the February issue.

Better Late Than Never

JAN 3 Emmet J. Duffy has responded.

JAN 4 Jeannette Roth has responded.

FEB 1 Eric Jamin has responded.

FEB 2 M. Fuerst claims that the hardest part is obtaining 129, 976, 320 as the number of complete closed routes over a heptagon with its diagonals. I agree.

FEB 3 Was proposed by Harry Nelson.

M/A 2 N. Peterson points out that the golden ratio is also the limit of

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$$

Forrest Meiere points out a small error in the published solution. The statement, "Any sequence of numbers formed according to the rule $U_{N+2} = U_{N+1} + U_N$ will exhibit the property that the

$\lim_{N\to\infty} U_{N+1}/U_N = (1 + \sqrt{5})/2$."

has one exception.

Try $U_N = \alpha^N$ with $\alpha = (1 - \sqrt{5})/2$. The general U_N is $cF_N + d\alpha^N$ for suitable c and d with your statement true if $c \neq 0$.

M/A 3 P. Bonomo and Elliot Roberts noticed that the formula for T also occurs in formulas for pendulums.

M/A 4 Winslow H. Hartford has responded.

M/A 5 Gregory James Ruffa has responded.

M/A SD 2 Robert Pogoff was misquoted in July/August. He did not claim that the published solution was correct only for low altitudes but rather that a simplification is possible for low altitudes. Dick Boyd also noticed this approximation, namely: the "horizontal distance" in miles is nearly 1¼ times the altitude in feet.

MAY 1 J. A. Faucher has responded.

MAY 2 Eric Jamin has responded.

MAY 3 Eric Jamin and Gregory James Ruffa have responded.

MAY 4 Eric Jamin has responded.

MAY 5 Eric Jamin, Harry Zantopulos, Gregory James Ruffa, and J. H. Meier have responded.

NS 2 Eric Jamin has responded.

NS 3 M. Fuerst noticed the solution in an issue of *Operations Research* last year.

Proposer's Solutions to Speed Problems

SD 1

L INK

I NCH

G ILL

H AND

T ON

Y ON

E LL

A CRE

R OD

And, if you care, a LIGHTYEAR = 9.4608×10^{17} cm.

SD 2 Joe bet that a pound of Brie cost \$4, and Jane bet it cost \$2.

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Letters

Continued from p. 4

Wiesner points out that efforts to protect society against the health and other *external* effects of new technologies inevitably slow the process of innovation. The rates of innovation to which we have become accustomed have been made possible, it seems, only by ignoring external costs. The more rapid the pace of innovation, the less likely we are to correctly assess the external costs: an obvious example is the 20-year lag in the development of cancer from human carcinogens. But even where such basic biological time clocks do not delay our realization of the full impact of change, learning takes time. By the time we notice the costs of change, the innovation is already embedded in the social structure (as with the automobile), or already being supplanted by the next innovation.

It is at least possible that slowing the pace of technological innovation will allow time for further external costs to emerge and join those already recognized. Note that the rejection on economic grounds of the first-generation supersonic transport provided time for scientists to discover still more undesirable side-effects.

It is not enough to say that various environmental and health problems are external costs imposed by specific, insufficiently analyzed technological changes; rather, they are costs of the pace of technological change itself. Since in the aggregate, society seems to have decided that these costs are excessive, efforts to restore the pace of innovation are mis-