Cross-metrics, Arithmocrypts, and a Tale of Two Runners

Puzzle Corner
by
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Welcome to "Puzzle Corner," starting its tenth volume in *Technology Review*!

For newcomers, here's how it works: Each month we publish five problems and several "speed problems," selected from those suggested by readers. The first selection each month will be either a bridge or a chess problem. We ask readers to send us their solutions to each problem, and three issues later we select for publication one of the answers — if any — to each problem; we also publish the names of other readers who submitted correct answers. Answers received too late or additional comments of special interest are published as space permits under "Better Late Than Never." And I cannot respond to readers' queries except through the column itself.

Please note especially this month's "Better Late Than Never" section, because a problem published in June, 1975, is modified and reissued in it. The solution to the modified problem will appear in the February, 1977, issue, together with those of the new problems presented in this column.

Please note also that the fifth problem presented in June, 1976, was incorrectly stated. It should read:

JUN 5 A man walking near a lake with a precipitous shoreline sees a girl struggling in the water. He can run twice as fast as *he* can swim. At what point should he leave the shore to reach her in the shortest possible time? The spatial relationships are: the man is 100 feet from the water; the distance between the man and girl, parallel to the shore line, is 100 feet; *and the girl is 100 feet from the shore.* Ground rule: no calculus. (The girl drowns while the man is calculating the best course, but that's irrelevant to the problem.)

Problems

Here are five new problems for this month. Solutions which reach me by December 1 will be in time for possible use in the February issue.

O/N 1 We begin with a bridge problem from *D-Notes*, a publication of the Charles Stark Draper Laboratories; *D-Notes'* bridge department is run by Ben Dores. Given the two hands and the bidding shown (both sides vulnerable), and the following first three plays, plan the balance of play. The opening lead was ♥3, won by East with ♥A; East returns ♥7 to West's ♥9; then West shifts to ♦3.

North
♠ J 7 3 2
♥ 6 2
♦ A K 8 7 6
♣ J 3

South
♠ A K 6 5 4
♥ 8 5
♦ 2
♣ A 10 6 5 4

South	*West*	*North*	*East*
1 club	—	1 diamond	1 heart
1 heart	3 hearts	3 spades	—
4 hearts	—	—	—

O/N 2 Magne Wathne noticed that 9/1 = 9, 98/12 = 8.166 . . ., and 987/123 = 8.024 . . . He then found patterns for the numerator and denominator which begin with the three given and result in fractions which approach 8. What are those patterns? Clearly {9, 98, 987, 8, 8, 8, . . .} and {1, 12, 123, 1, 1, 1, . . .} work, but we want a true pattern.

O/N 3 R. Robinson Rowe has submitted a cross-metric for us to tackle:

HHE ×	TEN =	AUITM
−	×	−
EALI −	HIE =	ETAM
=	=	H
CIIE +	ATMAM =	AHMLI

He explains that the cross-metric consists of nine literally-coded numbers related by six arithmetic operations, three being horizontal and three vertical. As coded, the numbers are words only by accident, but the code — in digital order — is literate and may be cited as the solution.

O/N 4 An interesting puzzle from Charles Piper: A offers to run three times around a course while B runs twice around, but A gets only 150 yards of his third round finished when B wins. He then offers to run four times around for B's thrice and now quickens his pace in the ratio of 4:3. B also quickens his in the ratio of 9:8 but in the second round falls back to his original pace of the first race and in the third round goes only nine yards for the ten he went in the first race, and accordingly this time A wins by 180 yards. Determine the length of the course.

O/N 5 We close with a measure theory problem from Frank Rubin; it looks tough: A sentential digital form S is a string of digits ? and ~ where ? represents any digit and ~ any digit string. A real number is S-bounded if S appears only finitely many times in its decimal expansion. Show that the S-bounded numbers have (Haar or Lebesque) measure O.

Example: 1?2~3 appears in
.351927536802 . . .
1?2~3

Speed Department

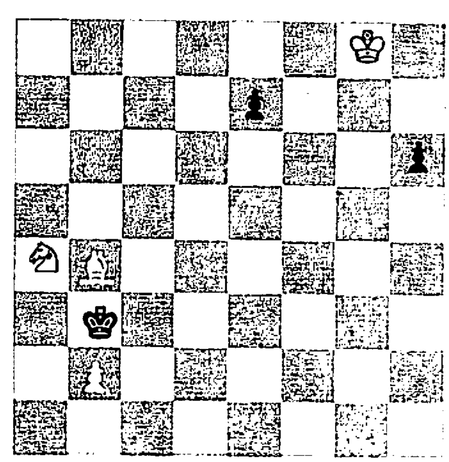
O/N SD 1 We have a novelty this month — a speed chess problem from Jeff Kenton; Mr. Kenton believes the problem originates with Sam Lloyd: Find a position in which a lone king is mated in the middle of the board by two rooks and a knight.

O/N SD 2 Our Lilliputian, Ted Mita, asks: If you were trapped under a giant phonograph record which way would you crawl to get out?

Solutions

The following are solutions to problems published in May:

MAY 1 Given the following, White is to play and win.



Basically, the responses fell into three camps: B–B3, B–Q2, and hopeless. I think B–Q2 is refuted by K x N, B x P, P–K4! The B–Q2 camp expected K–N6. The "hopeless" camp, which noticed P–K4!, missed P–N3 mate in the following solution from George Mortimer:

White can win in one of two ways:
— Capture the two black pawns while losing only one pawn and checkmate with K, B, and N vs. K.
— Queen his lone pawn.

From the given position, White cannot prevent loss of either N or B; therefore, the first option above is not under White's control. He must go for the second option. Since he must wage battle across the span of the board, he needs a piece which can span the board at a single move — the B. He must save it. Therefore, play follows:

	White	*Black*
1.	B — B3	K x N

White's move protects the P and the B by each other and prevents advance of Black's KP. Black now has only a K and RP to play with. Black could make many other moves, but if he did not take the N now it would be safely gone on White's second move and the game would no longer be a contest.

2.	K — B7	. . .

White's intention is to bring his K into position to guard his P and release his B to chase Black's Rs, since the White K cannot catch Black's RP. Black can stop this by playing K — N6, but then White is given his Black's RP and Black has nothing left to play with. Therefore,

2.	. . .	P — R4
3.	K — K6	P — R5
4.	K — Q5	P — R6

If black plays K — N6 in the fourth move instead, K — K4 in the fifth puts White's K even with Black's RP and he catches it as it queens. White's win is then routine.

5.	K — B4	. . .

Black's K is now immobilized. Perhaps he overlooks this fact in his mad race to obtain a queen. If so, then play continues:

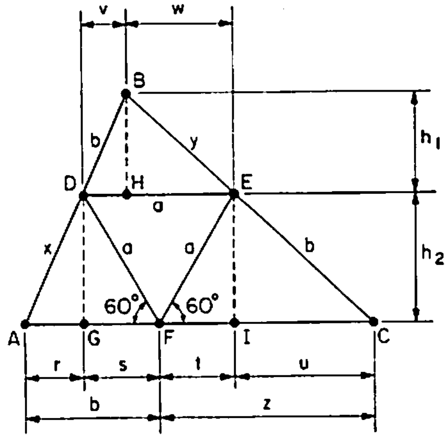
5.	. . .	P — R7
6.	B — N4	P — R8 (queens)
7.	P — N3 mate	

Responses received from Bill Munk, Richard Hess, Elliot Roberts, R. Bart, Edward Gaillard, Gerald Blum, William J. Butler, Jr., Mike Hill, and George Farnell.

MAY 2 Show that, given the equalities shown in the diagram at the top of the next column, triangle ABC is equilateral. The solution is from Harry Zaremba.

Assume AC is parallel to DE; then angles DFA and EFC = 60°. Thus $h_2 = a \sin 60 = \sqrt{3}a/2$, and $s = t = a \cos 60 = a/2$. For convenience, let $b = na$. Then $r = na - e = na - a/2 = a(2n - 1)/2$, and

$$x = (h_2^2 + r^2)^{1/2}, \text{ or}$$
$$x = a(n^2 - n + 1)^{1/2}. \qquad (1)$$



Similarly, $u = (b^2 - h_2^2)^{1/2}$, or $u = a(4n^2 - 3)^{1/2}/2$. Therefore,

$$z = t + u = a/2 + a(4n^2 - 3)^{1/2}/2. \qquad (2)$$

From similar triangles ADG and DBH, $h_1/na = h_2/x$, or $h_1 = \sqrt{3}na/2(n^2 - n + 1)^{1/2}$. Also, from similar triangles BHE and EIC, $y/h_1 = na/h_2$, or

$$y = n^2a/(n^2 - n + 1)^{1/2}. \qquad (3)$$

In triangle DBH, $v = [(na)^2 - h_1^2]^{1/2} = na(2n - 1)/2(n^2 - n + 1)^{1/2}$, and in triangle BHE, $w = (y^2 - h_1^2)^{1/2} = na(4n^2 - 3)^{1/2}/2(n^2 - n + 1)^{1/2}$. Hence $a = v + w$, or $a = [na(2n - 1) + na(4n^2 - 3)^{1/2}]/2(n^2 - n + 1)^{1/2}$. After removing radicals and simplifying,

$$3n^6 - 4n^5 + 3n^4 + n^3 - 4n^2 + 2n - 1 = 0 \qquad (4)$$

The rational solution of equation (4) is $n = 1$, making $b = na = a$. Substituting $n = 1$ into expressions (1), (2), and (3) results in $x = z = y = a$. Therefore, $b + x = b + y = b + z = 2a$, proving that triangle ABC is equilateral.

Also solved by R. Bart, Burnham H. Dodge, Mary Lindenberg, Jo Anne Levatin, Richard I. Hess, M. Fuerst, and R. Robinson Rowe.

MAY 3 A cryptarithmetic problem: which digit does each letter represent:

A BOY asked a GIRL to become his wife
When each one was in the prime of their
 life.
If they simply add LOVE, there is but
 one hope . . .
The result of it all is that they ELOPE.

The poem represents the mathematics problem:

```
  BOY
 GIRL
+ LOVE
 ELOPE
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with BOY and GIRL being primes.

The following is from William C. Schumacher, who says he was "fascinated to see the use of prime numbers as clues. For a simple appearing problem it required more deductions and more kinds of deductions than most cryptarithmetic problems (why not call them arithmocrypts?) that may look more complex on the surface." Here is Mr. Schumacher's solution:

a) B, G, L, and E, as the initial letters of the four words, are not zero.

b) The sum of three different digits (as B + I + O, or O + R + V, or Y + L + E) cannot exceed 24, and with a carry from another such column of three cannot exceed 26.

c) The sum of two different digits cannot exceed 17, and with a carry from another column of up to three different digits cannot exceed 19.

d) Therefore, E equals 1. (Representing the carry from the thousands column.)

e) Since "BOY" and "GIRL" are both primes, Y and L are members of the set (1, 3, 7, 9). But 1 is no longer available; therefore, they are members of the set (3, 7, 9).

f) From the units column, Y + L + E = E, or Y + L + E = E + 10. Therefore, Y and L are each other's complements modulo 10. But as neither of them can be 1 (because E = 1), so neither of them can be 9. Therefore Y and L are 3 and 7 (not necessarily respectively), so, while it is too early to specify which is which, no other letters can be 3 or 7.

g) From the hundreds column, B + I + O = O, or B + I + O = O ± 10, or B + I + O = O ± 20. As B and I cannot both be zero, the first possibility is ruled out; and as B ± I + (tens carry) cannot exceed 19, the third possibility is also ruled out. Therefore, B and I must be each other's complements modulo 10 if there is no tens carry, or modulo 9 or 8 if there is a tens carry of 1 or 2, respectively.

h) The thousands column G + L + (hundreds carry) = EL (or L + 10) requires that G be 8 or 9. But since B + I + O = O + 10, the hundreds carry must be 1, and G must be 9.

i) As the only uncommitted odd digit is 5, and the units carry is 1, the tens column O + R + V + (units carry) = P requires that one of these letters be 5, and the other three be even. In other words, O, R, V, and P are members of the set (O, 2, 4, 5, 6, 8) with the stipulation that one of them must be 5.

j) Examining the possible groupings of values to comply with the tens column addition O + R + V + 1 = P, or O + R + V + 1 = P + 10, it is found that all except one of such groupings lead to the sum P ± 10. The one that does not is O + 2 + 5 + 1 = 8.

k) If the tens column produces a carry of 1, B and I would have to be each other's complements modulo 9, which is impossible if they are limited (as they now are) to even digits. Therefore, the tens column must comprise digits 0, 2, and 5 as addends, 1 as incoming carry, and 8 as sum. With 8 no longer available, B and I must be 4 and 6, not necessarily respectively; but no other letters can be 4 or 6.

l) At this point, we are reduced to

B = 4 or 6
O = 0, 2, or 5
Y = 3 or 7
G = 9
I = 4 or 6
R = 0, 2, or 5
L = 3 or 7

Independently, each of these words could be assigned any of 12 values, but since the use of a digit in either denies it to the other, only 24 permissible conbinations exist. Moreover, by inspection five values in each group of twelve are multiples of 3 and two other values in each group are multiples of 7.

m) The remaining permissible values are

BOY = 403, 407, 457, 607, or 653
GIRL = 9403, 9407, 9427, 9607, or 9623

n) Of these, 403, 407, 9407, 9427, and 9607 are nonprimes. The only combination of primes with nonduplicating digits is 457/9623.

o) After assigning these digits to the appropriate letters, only V = O is left.

p) Checking the arithmetic, 457 + 9623 + 3501 = 13,581. Everything fits.

Also solved by Don Garvett, Erik Anderson, Jeffrey A. Miller, Charles Polay, Richard A. Bator, James D. Abbott, Mark Marinch, Paul Benefiel, Roger A. Whitman, Sam Wheatman, John D. Rothschild, Gerald Blum, William J. Butler, Jr., Harry Zaremba, R. Bart, Richard I. Hess, and the proposer, Avi Ornstein.

MAY 4 A four-legged stool stands on an uneven floor. There are no sudden steps, but the floor is wavy, with bumps and hollows. The stool will stand, of course, with three legs touching the floor. Is it always possible to turn and/or move the stool so that it stands firmly with *all four* legs touching the floor?

One thing must be made clear: the legs of the stool are of equal length and symmetrically placed. If the legs may be of unequal length, choose a stool with three very long legs and one very short; then if the floor is not too wavy the stool cannot have all legs touching. This analysis is from Richard I. Hess. With the equal-length assumption, the answer is yes. The following is from R. Robinson Rowe:

In a random setting of the stool, let the three legs touching the floor be A, B, and C, and let leg D be "airborne." Let the ground plane be one parallel to the one through the contact points of legs A, B, and C, below that plane and above the floor under leg D. Then, with respect to this ground plane, legs A, B, and C are at high points and leg D is above a low point. In any self-respecting stool, the four legs are in symmetry such that their feet may be inscribed in a circle. Project that circle on the undulating floor and elevations along the perimeter, if developed, would graph as an irregular sinuous curve — partly above and partly below the gound plane. That curve has no "steps." Now rotate the stool about its vertical axis, keeping legs A, B, and C in contact with the floor. If rotation reached 90°, leg D would be at a high point and leg C *over* the established low point — and no longer in contact. There being no sudden steps, at some intermediate point, D would have made contact and C would still be in contact. Hence it is always possible to *turn* the stool to a stable setting. I use this strategem often. To replace a bulb in a ceiling electrolier, I use a kitchen stool and for safety rotate it until it is steady.

Also solved by R. Bart, Gerald Blum, and William J. Butler, Jr.

MAY 5 A square number is one which can be represented by an array of points in the form of a square. Similarly, a triangular number is one which can be represented by an array of points in the form of an equilateral triangle. The square numbers are 1, 4, 9, 16, 25, 36, . . . The triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, . . . The first number after 1 which is both a square and a triangular number is 36; what is the next number which is both square and triangular?

Several readers gave detailed analyses for this problem — far beyond what was called for. In the interest of space economy, I am presenting a somewhat terse account from Winslow H. Hartford: Since triangular numbers are represented by $n(n + 1)/2$ and squares by m^2, we require a solution of the equation $n^2 + n = 2m^2$. Solutions exist when n = 1, m = 1 and n = 8, m = 6. However, there are two cases:

— $n + 1 = 2k^2$, $n = l^2$ ($m^2 = k^2l^2$) where $l^2 + 1 = 2k^2$ or $2k^2 - l^2 = 1$.

— $n + 1 = l^2$, $n = 2k^2$, where $2k^2 + 1 = l^2$ or $2k^2 - l^2 = -1$ ($2k^2 - l^2 = \pm 1$).

It may be shown that solutions exist where

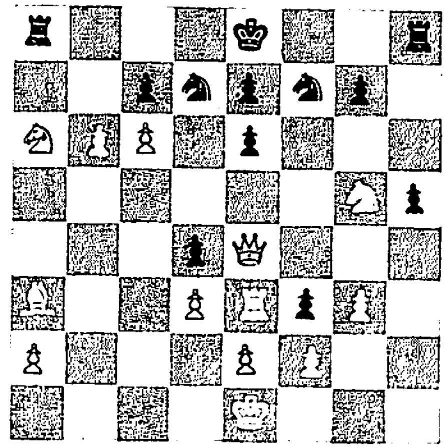
	k	l	n	n + 1	m	$m^2 = \Delta$
(2)	0	1	0	1	0	0
(1)	1	1	1	1	1	1
(2)	2	3	8	9	6	36
(*)	5	7	49	50	35	*1225*
(2)	12	17	288	289	204	41616
(1)	29	41	1681	1682	1189	1413721
(2)	70	99	9800	9801	6930	48024900

The asterisk marks the requested solution. It is also obvious that l/k forms a Fibonacci-like series: 1/0, 1/1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169 . . . where $l_{n+1} = l_n + 2k_n$ and $k_{n+1} = l_n + k_n$, and $\lim_{n\to\infty} = l/k = \sqrt{2}$.

Also solved by Bob Schmidt, Vern Reisenleiter, John E. Prussing, Carl F. Muckenhoupt, Frank Carbin, John Colton, R. Robinson Rowe, Johan Novick, Naomi Markovitz, R. E. Crandall, Charles Rozier, Nadin Godrej, Sanford Libman, Raymond Gaillard, Emmet Duffy, Mark Marinch, Dick Boyd, James Fiasconaro, George Mortimer, James Cawse, Gerald Blum, Robert Bart, Erik Anderson, Thomas Jenkins, Roger Whitman, Paul Benefiel, Jeffrey Miller, Richard Hess, Harry Zaremba, Sam Whentman, Avi Ornstein, Mary Lindenberg, William J. Butler, Jr., and the proposer, Kier Finlayson.

Better Late Than Never

1975 JUN 1 Harry Nelson, the proposer, informs me that the White pawn at KR5 should be Black. He claims the problem is now possible; I can come close, but . . . The correct problem is: White to move and mate in two.



Thus the problem is reinstated, and a solution will be given in February.

1975 O/N 1 A rebuttal from William J. Butler, Jr.: I originally sent in a table of combinations for this problem, and this table was published in February, 1976. The June, 1976, issue contains another analysis which disagrees with this table. I believe there are two flaws in the June analysis which accounts for the difference. First, in calculating probabilities either all possible outcomes must be equally likely or else the outcomes must be proportionately weighted. As soon as a player makes a voluntary discard he alters random probability. For example if West started with no spades (three diamonds), eight clubs, and two hearts, he would deliberately avoid discarding a heart and thus alter random results. If West discarded randomly on the spade tricks and was lucky enough not to throw away a heart, then the analysis given in the June issue would be proper. The second flaw in the June analysis is the conclusion that South learns something about the original club-heart distribution while he plays the spades. In fact he only learns the spade distribution. For example, consider these

Number of families	25		50		25	
	Boys	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*
P distribution of offspring	40	60	50	50	60	40
Number of offspring	25	38.5	50	50	25	16.7
Overall ratio			100	104.2		

one. In the real world, at least, a predictable feature of large human populations is that for every 41 births, the odd spare is a male. To solve the problem, it seems one should recognize two populations, the Mortals and the Immortals. The Immortals are members of an even numbered set, since people enter by pairs when they marry and leave by pairs when their sons are born. Therefore, the male-female ratio is one to one for the Immortal population. For the Mortal set the male-female ratio would no doubt begin at 21:20 but would soon be different, with the Immortals sending extra daughters, begat from their Sisyphean labor in trying to get a son. With an ample supply of mates for the spare males, a population increase in both sets would seem initially imminent. Over a vast expanse of time, would all males become Immortal from gradually marrying sterile and daughter-only mates? Would the adult Mortal population become an endless line of nubile maidens, wandering toward the lunch counter to commit suicide? One feels that the above is a likely outcome if the rules are followed as given.

Proposers' Solutions to Speed Problems

SD 1

	R	K	R	N	

SD 2 Counterclockwise, as viewed from above.

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