Returning to the Infinite Jail

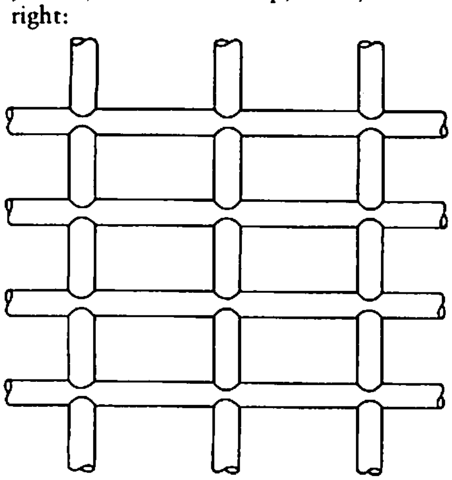
Puzzle Corner
by
Allan J. Gottlieb

Hello again. Although it is only the middle of May, I know that through the miracles of modern communication you think it is already August. These miracles partially account for the delay in seeing your problems in print. The greater delay, however, is that I have a fairly large backlog of problems. For this I am grateful. Let me be more specific, and thereby answer a number of reader inquiries: I have a large and growing supply of regular problems — well more than a year's worth. When added to the above-mentioned miracles, this means that any problems sent in today will not appear until 1978! For bridge and chess problems, however, the prospects are better — the backlog is just a few months. And "speed" problems — especially good ones — are in short supply.

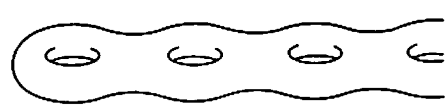
We now have a second permanent problem; please see the solution to M/A 5, below, for details. And we continue the policy of presenting previously-unsolved problems — this time a problem explained to me by Mike Spivac (of *Calculus on Manifolds* and *Calculus Calculus Calculus Calculus Calculus Calculus Calculus Calculus* fame). It's shown below as NS 4, first published in February, 1969, as the 16th problem in Puzzle Corner for that volume of the *Review*.

Problems

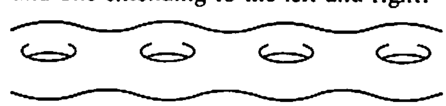
NS 4 Consider the surface of an infinite jail cell, which extends up, down, left and right:



and two infinite holed tori, one extending to the right:



and one extending to the left and right:



Are any two of these three homeomorphic? Why, or why not?

J/A 1 Our first new offering is a nine-part chess problem from Thomas O. Mahon, Jr.:

What is the minimum number of pieces required for a position in which

A. If White is to move,
1. The situation is a stalemate;
2. White must win;
3. Black must win. Or

B. If Black is to move,
1. The situation is a stalemate;
2. White must win;
3. Black must win.

The three requirements when White is to move combine with the requirements when Black is to move to create nine sub-problems. Two comments are in order: pieces include pawns; and the sub-problem A = 2, B = 3, for example, requires that when White moves first he wins for any sequence of legal moves, and similarly for Black.

J/A 2 Our second problem is from Irving L. Hopkins:

It is well known that Pythagorean triples with integers a, b, h, where $a^2 + b^2 = h^2$, may be generated by taking $h = p^2 + q^2$, $a = p^2 - q^2$, and $b = 2pq$, where p and q are any integers such that $p > q \geq 1$. A special series of such triples is that in which the legs a and b differ by one unit. By means of a table of squares we can find that the first four of such triples are:

p	q	h	a	b
2	1	5	3	4
5	2	29	21	20
12	5	169	119	120
29	12	985	697	696

(For the tenth such triple, p = 5741, q = 2378, h = 38,613,965, a = 27,304,197, and b = 27,304,196.) If we take the fractions q/p from the table above, they are 1/2, 2/5, 5/12, and 12/29. These are the continued fraction approximants for $\sqrt{2} - 1$. Prove that all the approximants (a) result in Pythagorean triples (b) with legs differing by unity, and (c) that the dimensions increase nearly geometrically with a ratio approaching $(3 + 2\sqrt{2})$.

J/A 3 Robert Pogoff wants a straightedge-and-compass construction of a triangle given the lengths of the three medians.

J/A 4 Here is a number theoretic problem from Emmet J. Duffy:

Write down in a row the coefficients of the expansion of $(a + b)^N$ where N is any integer. Multiply each coefficient by the second coefficient and set the result below the multiplied coefficient. Continue by multiplying by the third, fourth, and fifth coefficients, etc. An example is shown for the coefficients of the expansion of $(a + b)^4$:

1	4	6	4	1
4	16	24	16	4
6	24	36	24	6
4	16	24	16	4
1	4	6	4	1

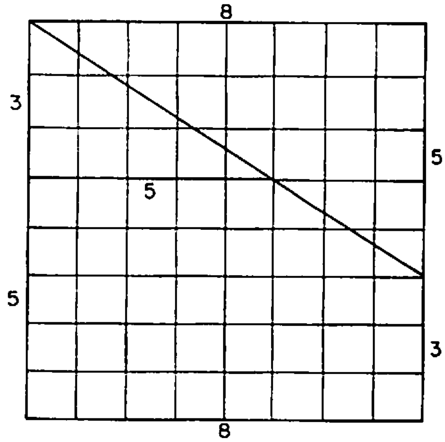
Find a formula for the sum of the digits in any diagonal.

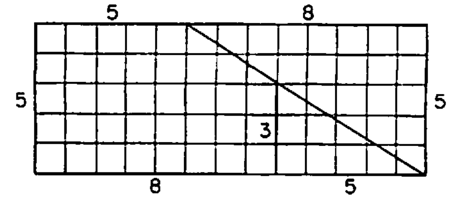
J/A 5 Stuart Schulman has supplied our last problem — a logic problem. It was brought to mind, he says, by the preface to a George Gamow book read many years ago. Gamow's problem involved three people who wanted to split a decanter of wine equally but had no measuring devices. Of course, with just two the solution is simple: one of them divides the wine until he/she is satisfied with either portion, and then the other is allowed to choose. With three people it gets a bit more complicated; Schulman's problem: Can the question be generalized to a larger number of people, by finding a way to divide the wine for any number so that each is satisfied that his/her share is (at least) a fair one? Assume that each person is willing to divide the wine and accept any of the portions.

Speed Department

J/A SD 1 P. V. Heftler wants you to find three primes A, B, and C such that $A^2 + B^2 = C^2$.

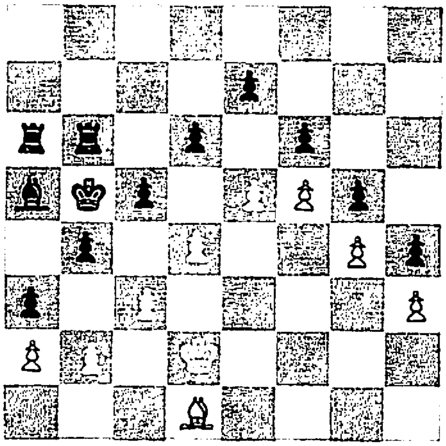
J/A SD 2 Winthrop Leeds wants you to explain how 64 units can be divided into three pieces which can be reassembled into 65 units (*see right*):





Solutions

DEC 1 (as corrected in the March/April issue) White to play and draw:



The following solution is from S. J. Warner and Virginia S. Glessner:

1. B — R4	ch	K × B*
2. P — N3	ch	K — N4
3. P — B4	ch	K — B3
4. P — Q5	ch	any
5. P — K6		

The above results in a pawn blockade which cannot be penetrated; thus no further captures can be made.

* If K — N5, then 2. B — N3 ch, etc., resulting in a perpetual check.

Also solved by William J. Butler, Jr., A. Le Blanc, Abe Schwartz, Herbert R. Moeller, Rony Adelsman, Gerald Blum, Jerome J. Taylor, Ralph Menikoff, Ralph Wagner, and the proposer, Mike G. Middlebrooke.

M/A 1 South is declarer at a contract of six spades. How can he make the contract after West leads ♥ K?

North:
♠ K Q 3
♥ –
♦ Q 8 3
♣ A K 9 8 6 4 2

West:
♠ J 10 8
♥ A K 6 4
♦ K J 7 4
♣ 7 3

East:
♠ 6
♥ 9 8 7 5 3 2
♦ 10 9 2
♣ Q *J 10*

South:
♠ A 9 7 5 4 2
♥ Q J 10
♦ A 6 5
♣ 5

Most readers noticed that in the problem as originally published East needed the ♣J and ♣10, added in italics above. Albert J. Fisher has submitted a solution which includes a few possible pitfalls:

The secret of my solution lies in the realization that North's clubs must be established and that no end-play or throw-in against West can succeed if both West and South have hearts in hand when West is put on lead. Then it becomes clear that the whole problem is one of not letting the defense in too soon. All the schemes involving ruffing the third club low or pitching a heart suffer from this defect — the defense is in a position to make a damaging return which prevents South from drawing trumps and enjoying the established clubs. The way to prevent this is to set up the clubs by ruffing the third round with ♠ A. The play goes as follows:

1. Ruff low
2. ♣A
3. ♣K, discarding a heart
4. Small club, ruffing with the ♠A.
5. Trump to the ♠K.
6. Small established club, discarding the last heart from South.

The position prior to the lead of trick 6. is:

North:
♠ K
♥ – – –
♦ Q x x
♣ x x x x

West:
♠ J 10
♥ A x x
♦ K J x
♣ –

East:
♠ – – –
♥ 9 x x x x
♦ 10 9 x
♣ –

South:
♠ 9 x x x
♥ Q
♦ A x x
♣ – –

By ruffing with the ♠A, South has established a trump trick for West. In return, however, he has gained a tempo: when South leads his good club and pitches his remaining heart, West is well and truly fixed. He can ruff in, but South can handle any red suit return, while the remaining spade in dummy serves the dual purpose of extracting the outstanding trump and providing entry to the rest of the club suit. If West elects to ruff in but throws off red cards, South rids himself of his losing diamonds, cashes his ♦A, and concedes one trump trick. One interesting feature of the hand is how few high cards North and South really need in order to fulfill this slam. If the hands had been

North:
♠ K Q x
♥ –
♦ x x x
♣ A K x x x x x

West:
♠ J 10 9
♥ A K Q J
♦ K Q J 10
♣ Q J

East:
♠ x
♥ x x x x x x
♦ x x x
♣ x x x

South:
♠ A x x x x x
♥ x x x
♦ A x x
♣ x

the evolution of the spade slam would proceed exactly as described above, unbeatable against any defense.

Also solved by Elmer C. Ingraham, William J. Butler, Jr., Jerrold Grossman, Jon Froemke, F. F. Schultz, R. Robinson Rowe, John Rollino, Avi Ornstein, Rick Collarini, Carl S. Kelley, and the proposer, Russell A. Nahigian.

M/A 2 Recall the Fibbonacci numbers defined by: $F_1 = F_2 = 1$ and $F_{N+2} = F_N + F_{N+1}$ for $N \geq 1$. This sequence begins 1,1,2,3,5,8,... The problem is to prove that

$$\lim_{N\to\infty} F_{N+1}/F_N = (1 + \sqrt{5})/2.$$

Some people "cheated" by assuming that the limit exists and then calculating its value using the recursive formula for Fibbonacci numbers. The hardest part, of course, is to show that the limit does exist. Nonetheless, a fine solution was submitted by Stephen Speicher:

Notice that expanding $x/(-x^2 - x + 1)$ leads to

$$x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \ldots + F_N x^N,$$

where the coefficient of x^N is F_N, the Nth Fibbonacci number. The general term for F_N, the coefficient of x^N, may be found as follows: Use partial fractions to express:

$$x/(-x^2 - x + 1) = A/(x - a) + B/(x - b), \quad (1)$$

where a and b are the roots of the equation

$$-x^2 - x + 1 = 0; \; a = -(1 + \sqrt{5})/2, \; b = -(1 - \sqrt{5})/2.$$

Then $x = A(x - b) + B(x - a)$, and by equating coefficients determine that $A = -a/(b - a)$, $B = b/(b - a)$. Substituting these results into (1) and rearranging terms, the expression becomes

$$1/(b - a)[1/(1 - x/a) - 1/(1 - x/b)].$$

This term may be expressed as

$$1/(b - a)[(1 + x/a + (x/a)^2 + \ldots + (x/a)^N] - [1 + x/b + (x/b)^2 + \ldots + (x/b)^N].$$

This expression now shows that the

coefficient of x^N, which is F_N of the original expansion, is $F_N = 1/(b - a)(1/a^N - 1/b^N) = 1/(b - a) \cdot (b^N - a^N)/a^N b^N$. But $ab = (-1)(1 + \sqrt{5})/2(-1) \cdot (1 - \sqrt{5})/2 = -1$; therefore $a^N b^N = (-1)^N$, and then $F_N = (-1)^N/(b - a) \cdot (b^N - a^N)$.

$$\text{Now } \frac{F_{N+1}}{F_N} = \frac{(-1)^{N+1}/(b-a) \cdot (b^{N+1} - a^{N+1})}{(-1)^N/(b-a) \cdot (b^N - a^N)}$$
$$= -\left[\frac{b^{N+1} - a^{N+1}}{b^N - a^N}\right].$$

However, since $b = -(1 - \sqrt{5})/2$ which is < 1, then the

$\lim_{N\to\infty} b^N = 0$. Therefore

$$F_{N+1}/F_N = -(-a^{N+1}/-a^N) = -a$$
$$= -(-1)(1 + \sqrt{5})/2 = (1 + \sqrt{5})/2. \quad \text{Q.E.D.}$$

Some additional points are worth noting. The number $(1 + \sqrt{5})/2$ is, of course, the legendary golden ratio, used extensively by the Greeks in the proportioning of their artifacts and architecture, the best example being the Parthenon. This golden ratio, $1 : (1 + \sqrt{5})/2$, is seen in such simple figures as the sides of a golden rectangle, as well as a generator for more complex structures such as a logarithmic spiral. To elaborate on the use of the golden ratio and its relationship to natural phenomena could easily fill several volumes. More appropriate to M/A 2, however, is the realization that there was nothing unique in the choice of the Fibbonacci sequence leading to the limit $(1 + \sqrt{5})/2$. Your readers might enjoy noting that any sequence of numbers formed according to the rule $U_{N+2} = U_{N+1} + U_N$ will exhibit the property that the

$$\lim_{N\to\infty} U_{N+1}/U_N = (1 + \sqrt{5})/2.$$

The choice of the initiating sequence as being 1,1 (the Fibbonacci generators) was entirely arbitrary. Using −5,2 or 3,−1, etc., would produce the same limit as $N \to \infty$. One final note: It is interesting to see that the simplest of continued fractions

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}$$

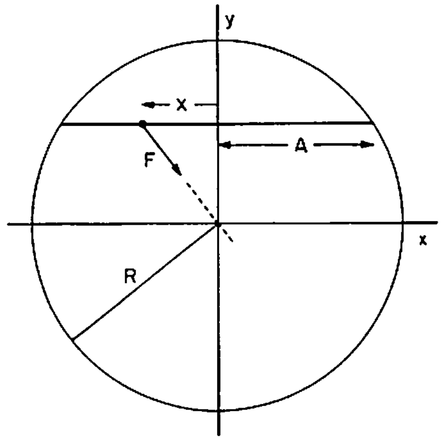
also has as a limit $(1 + \sqrt{5})/2$. This can be seen by forming successive convergents 1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, 34/21, . . . , which is easily recognized as being F_{N+1}/F_N.

Solutions also received from Mark Marinch, Hillary Fisher, Charles Gordon, R. Robinson Rowe, William J. Butler, Jr., Stephen P. Hirshman, Hank Lieberman, Peter M. Silverberg, Robert H. Bicker, K. Haruta, Arthur W. Anderson, Harry Zaremba, Gerald Blum, Frank Carbin, Philip O. Martel, and the proposer, John Prussing.

M/A 3 A frictionless train runs by gravity in a straight tunnel between two points on the earth's surface. Find the maximum velocity and the time for a round trip. Show that the latter is independent of the length of the tunnel.

The following is from Richard Hanau:

Let the length of the tunnel be 2A. Without loss of generality, orient the coordinate system so that one axis bisects the tunnel. If we assume the density of the earth to be constant, independent of position, Gauss' Law will show that the gravitational field inside the earth is proportional to r, the distance from the center. Hence the gravitational force on the train is F = kr. The component of this force along the tunnel is kx, always directed toward the midpoint.



Because $F_x = -kx$, the motion is simple harmonic motion with amplitude A, and the equations of simple harmonic motion can be used. (This assumes the train starts at the surface of the earth with zero velocity.) The maximum velocity, which occurs at the midpoint, can be found by equating the total energy at the midpoint (all kinetic), $mV^2_{max}/2$, to the total energy at the surface (all potential), $kA^2/2$. (The motion is frictionless.) Hence $V_{max} = A\sqrt{k/m}$. To evaluate k, note that at the earth's surface, where the acceleration due to gravity is g, the gravitational force is mg = kR. So k = mg/R, and $V_{max} = A\sqrt{g/R}$. Hence all trains, of different masses, have the same maximum velocity — indeed, the same motion — in a given tunnel. The time for a round trip is the period, T, of the simple harmonic motion. Since $V_{max} = 2\pi A/T$, we have $T = 2\pi\sqrt{R/g}$, independent of the length of the tunnel.

Also solved by F. N. Steigman, Emmet J. Duffy, James P. Ballard, Robert Pogoff, William J. Butler, Jr., Stephen P. Hirshman, Arthur W. Anderson, Harry Zaremba, Gerald Blum, and the proposer, Jack Parsons.

M/A 4 A palindrome is a word or a sentence which spells the same thing when spelled backwards — "rotator," for example, or "Madam, I'm Adam." What about a word to describe a word (or a sentence) which spells another word (or a sentence) when spelled backwards — for example, "devil, sung"?

I present everyone's solutions together: "rewarder," "deveiled," "reflowed," "deflower," "retooler," and "deserts" are eight letters. The winner is "deliverer" (nine letters). The solvers were Joyce Tang (a "drow"), Albert J. Gracia, Rick Collarini, Avi Ornstein, Peter M. Silverberg, Harry Zaremba, and R. Robinson Rowe (who also proposed "beilttog").

M/A 5 Construct the integers from 1 to 30 using four 4s. For example, $14 = \sqrt{4} + 4 + 4 + 4$.

Due to suggestions from several readers, this problem is hereby declared PERM 2. Of course the limit of 30 is dropped. Please note that the greatest integer function is not allowed.

The solution to the original problem is from Frank Carbin and is shown in the box at the bottom of this page.

Also solved by Gerald Blum, Harry Zaremba, F. N. Steigman, Rick Collarini, Philip O. Martel, Avi Ornstein, John Rollino, Donald Zalkin, Richard Williams, Mark A. Frahlman, Naomi Markovitz, Wendy Elane Erb, J. D. Miller, Stuart D. Casper, William J. Butler, Jr., George Ropes, Peter M. Silverberg, John Kavazanjian, Paul Manoogian, David Finkel, William G. Hutchison, Jr., and the proposer, Bill Saidell.

Better Late Than Never

O/N SD 1 Clifton N. Lovenberg and Joan Young point out that Mr. Horvitz's solution should be amended to say that one quarter of the children of Aa children will be aa.

DEC 2 J. David R. Kramer has solutions for K = 5 and K = 13. Can anyone find a procedure for all K (or at least for infinitely many)?

DEC 4 Dr. Prussing points out a typographical error in his printed solution. Move "]/6" from the second line of his formula to the end of the third line.

DEC 5 R. Robinson Rowe has improved on the published solution:

The task is accomplished by establishing depots at way points for caches of fuel in 25-gallon units. If *n* units are cached at the first depot, one unit is used to deliver n − 1 units to the next depot. This requires

1. (4 × 4)/(4 × 4)	11. (4! + 4! − 4)/4	21. (4.4 + 4)/.4
2. 4/4 + 4/4	12. (44 + 4)/4	22. 4! − (4 + 4)/4
3. (4 × 4 − 4)/4	13. (4! + 4! + 4)/4	23. (4 × 4! − 4)/4
4. 4 + (4 − 4)/4	14. −.4 + 4 × (4 − .4)	24. 4! + (4 − 4)/4
5. (4 × 4 + 4)/4	15. 4 × 4 − 4/4	25. (4 × 4! + 4)/4
6. (4! + 4 − 4)/4	16. $4^4/(4 \times 4)$	26. 4! + (4 + 4)/4
7. 4 + 4 − 4/4	17. 4 × 4 + 4/4	27. 4! + 4 − 4/4
8. (4! + 4 + 4)/4	18. (4 × 4! − 4!)/4	28. 4 × (4! + 4)/4
9. 4 + 4 + 4/4	19. 4! − 4 − 4/4	29. 4! + 4 + 4/4
10. (44 − 4)/4	20. 4! − 4 + 4 − 4	30. (4 × 4! + 4!)/4

n − 1 round trips and a final one-way trip aggregating 250 miles, so the leg between depots is 250/(2n − 1) miles. The next leg will be 250/(2n − 3) miles, and the system continues until a final leg of 250/1 miles reaches the goal at Mile 1000. The total distance covered is

$$L = 250\left(\frac{1}{2n-1} + \frac{1}{2n-3} + \frac{1}{2n-5} + \ldots + \frac{1}{1}\right)$$

$$= 250\sum_{i=1}^{n} \frac{1}{2i-1} = 250\,\phi(n).$$

Now L = 1000 if $\phi(n) = 4$, but since $\phi(418) = 3{,}999\ 495\ 848\ 5098\ldots$ and $\phi(419) = 4.000\ 690\ 591\ 6401\ldots$, the problem requires 418 ideal legs as above and one fractional leg.

The published solution put this fractional leg last and consumed 10474.982 735 gallons of fuel. The best strategy is to put the fractional leg first, with a length of $L_0 = 1000 - 250\,\phi(418) = 0.126\ 037\ 8726$ miles.

To deliver 418 units of fuel to the first depot, there will be 418 round trips and a final one-way trip, consuming fuel for $837L_0$ miles. Thus the total fuel consumption is

$$\begin{aligned} G &= 418 \times 25 + 837L_0/10 \\ &= 10450 + 837[100 - 25\phi(418)] \\ &= 94150 - 20925\,\phi(418) \\ &= 10460.549\ 369\ 932\ 44 \text{ gallons,} \end{aligned}$$

Continued on p. 72

which is nearly 15 gallons less than the published solution.

Mr. Rowe has appended a demonstration of the computation of $\phi(n)$, for which space is not available. Readers may obtain copies on request from the Editors of the Review, Room E19-430, M.I.T., Cambridge, Mass., 02139.

M/A SD 2 Robert Pogoff points out that the given solution is good only for low altitudes.

The following have responded to the problems indicated:

PERM 1 Gerald Blum.
NS 2 A. C. Lawson and Bruce Simon.
O/N 4 Hans P. Lieb, Frederick Smalkin (a farmer, most appropriately), and John Young.
JAN 2 N. Spencer.
JAN 4 A. W. Collins.
FEB 2 Neil Hopkins.
FEB 4 Hans P. Leib.

Proposers' Solutions to Speed Problems

J/A SD 1 Mr. Hoftler did not supply a solution, so I will tell you why none is possible. We must have either three even numbers or one even and two odd numbers. Since $2^2 + 2^2 \neq 2^2$, we are reduced to $A^2 + 4 = C^2$ or $A^2 + B^2 = 4$ with A, B, and C odd. The latter is clearly impossible, so consider the former. Let $A = 2N + 1$ and $B = 2M + 1$. We require $(2N + 1)^2 + 4 = (2M + 1)^2$; i.e., $N^2 + N + 1 = M^2 + M$, which is impossible (M must be greater than N but $M = N + 1$ is too big).

J/A SD 2 The picture is a beautiful cheat! The grid is a little warped and the large triangles are really (noncongruent) quadrilaterals.

Allan J. Gottlieb is Coordinator of Computer Activities at York College of the City University of New York. Send Problems, solutions, and comments to him at York College, 150-14 Jamaica Avenue, Jamaica, N.Y., 11451.

Letters

Continued from p. 3

dynamics model, is available as a basis for study and for use in choosing among alternative policies. The human and economic penalty for unwise governmental policies is potentially so staggering that the country can ill afford to "muddle through." Just concerns about dynamic models of social and economic behavior should not cast doubt on our ability to deal with social complexity at a time when public frustration and uncertainty are increasing. Instead, the opportunity to reach a better understanding of social problems should spur an accelerated effort to improve models for clarifying economic change and for evaluating alternative policies.

Classified