Philately, Cryptarithmetic, and a Three-Legged Stool

Puzzle Corner
by
Allan J. Gottlieb

When we read the January issue, R. Robinson Rowe and I were surprised to find no speed problems. I referred to my original manuscript, and — sure enough — just plain forgot them. Sorry.

York College now has its own zip code, and I have a new position as you may have noted in the March/April issue. My job is now Coordinator of Computer Activity, and the zip is 11451; the complete address appears, as usual, at the end of the column. I guess I have to admit that, with the New York City budget crisis, my address might change again soon?

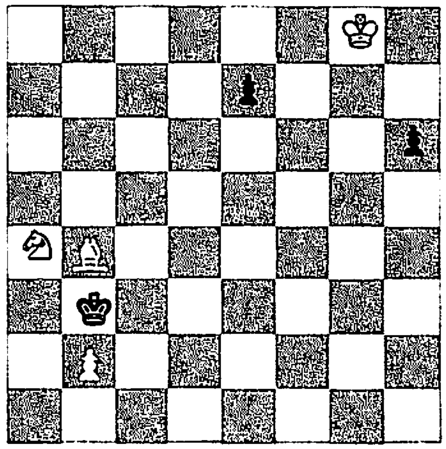
John Melchiori, a colleague of mine at York College, and I are writing a computer program to play chess, hoping to enter it in the North American championships in October. The program is named ALMA after our wives, *Al*ice and *Ma*ria, who are sacrificing Saturdays for the cause. Perhaps, soon it will be able to solve "Puzzle Corner" chess problems; currently ALMA finds any mate in one. Any advice would be welcome.

Problems

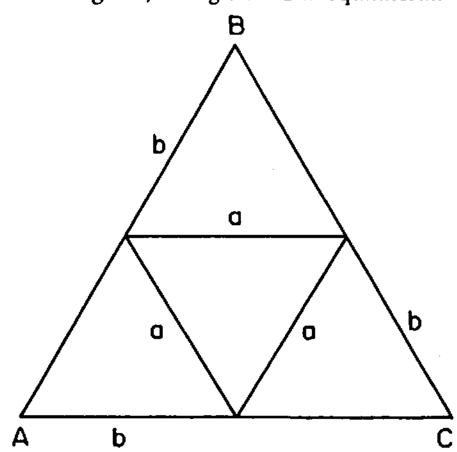
As you may recall, in December we began a policy of presenting old problems which were never completely solved. The following "stamp problem" appeared the month of my M.I.T. graduation, June, 1967. It doesn't seem that bad.

NS3 Suppose the government wants to revise its philatelic system and create only seven denominations of stamps. To aid in automatic processing, a maximum of only three stamps is to be permitted on an envelope (two or one are also permitted). Up to what value of postage will this cover without a break? What are the denominations of the stamps required to achieve this? When the problem first appeared, Richard L. Heimer solved it using a "brute force technique" on a digital computer; his answer: 70 cents using stamps of 1¢, 4¢, 5¢, 15¢, 18¢, 27¢, and 34¢. But he was troubled because he could not solve problems of this general class without resorting to the "brute force" method, and we still wait for such a solution.

MAY 1 We begin with an endgame problem from Harry Nelson. Given the following, White is to play and win.



MAY 2 Robert Kimble, Jr., wants you to show that, given the equalities shown in the diagram, triangle ABC is equilateral.



MAY 3 Avi Ornstein says the following is a cryptarithmetic problem, a kind he's not previously seen in "Puzzle Corner" in three years of reading. For those who are not introduced, a cryptarithmetic problem has letters in place of digits. The problem is resolving which digit each letter represents:

A BOY asked a GIRL to become his wife
When each one was in the prime of their life.
If they simply add LOVE, there is but one hope . . .
The result of it all is that they ELOPE.

The poem represents the mathematics problem:

```
   BOY
  GIRL
+ LOVE
 ELOPE
```

with BOY and GIRL being primes.

Mr. Ornstein notes that he originally concocted this problem in high school, but it had many solutions. By requiring that BOY and GIRL be primes, the problem is limited to one solution. (Cryptarithmetic problems have appeared in the past, but rather infrequently. — Ed.)

MAY 4 Dan Fingerman attributes the following problem to George Pillai: A four-legged stool stands on an uneven floor. There are no sudden steps, but the floor is wavy, with bumps and hollows. The stool will stand, of course, with three legs touching the floor. Is it always possible to turn and/or move the stool so that it stands firmly with *all four* legs touching the floor?

MAY 5 Our final regular problem is a fifth-grade homework problem which Kier Finlayson modified for *Technology Review*: A square number is one which can be represented by an array of points in the form of a square. Similarly, a triangular number is one which can be represented by an array of points in the form of an equilateral triangle. The square numbers are 1, 4, 9, 16, 25, 36, . . . The triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, . . . The first number after 1 which is both a square and a triangular number is 36; what is the next number which is both square and triangular?

Speed Department

MAY SD 1 We begin with a selection from John Rule: A three-volume set of books, each book 100 pages long, is placed on a bookshelf in the usual order, from left to right: volume 1, volume 2, and volume 3. The covers are all ⅛ inch thick; 100 pages are 1 inch thick. A worm starts between the front cover and page 1 of volume 1 and eats perpendicularly into the books until he is between page 300 and the back cover of volume 3. How far does he eat?

MAY SD 2 A nautical quicky from Emmet J. Duffy: A vessel is anchored three miles offshore, and opposite a point five miles farther along the shore another vessel is anchored nine miles from the shore. A boat from the first vessel is to land a passenger on the shore and then proceed to the other vessel. What is the shortest course for the boat? (No calculus, please.)

Solutions

The following are solutions to problems published in *Technology Review* for January.

JAN 1 With the following hands, clubs are trump and North is to lead.

	♠ A 2 ♥ A 2 ♦ A 4 3 2 ♣ —	
♠ K Q ♥ K Q ♦ K Q J 10 ♣ —		♠ J 10 9 ♥ J 10 9 ♦ 9 8 ♣ —
	♠ 4 3 ♥ 4 3 ♦ — ♣ 5 4 3 2	

The problem: North and South to take all eight tricks against any defense.

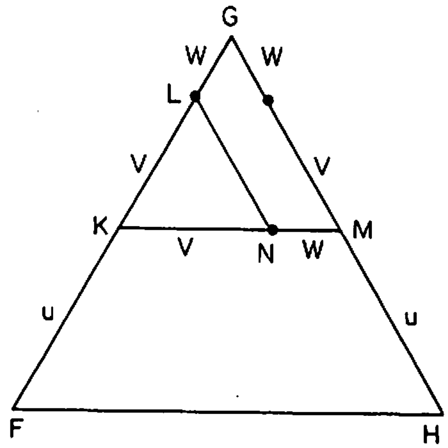
J. C. Kingery writes that he enjoys "Puzzle Corner" but is "rarely able to find a solution." But he was able to untangle this one: 1. North leads a low diamond, South ruffs. 2. South leads trump; West must hold diamonds to avoid establishing North's fourth diamond; if he discards a heart, North discards a low heart; East will discard a diamond. 3. South leads a heart, North takes the trick with ♥ A. 4. North leads ♦ A, South discards a low spade. 5. North leads a low diamond and South ruffs. The distribution is now as follows:

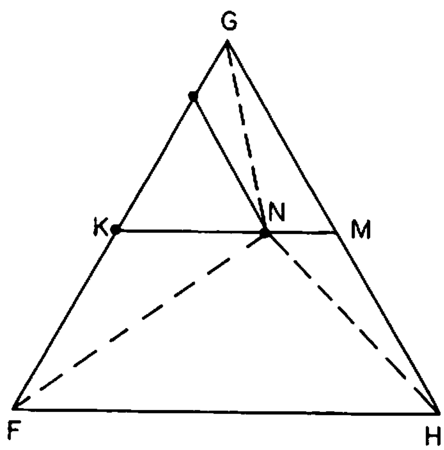
	♠ A 2 ♥ — ♦ 2 ♣ —	
♠ K Q ♥ — ♦ K ♣ —		♠ J 10 ♥ J ♦ — ♣ —
	♠ 3 ♥ 3 ♦ — ♣ 3	

If East is holding any other combination, spades or hearts will already be set up. 6. South leads the final club; if West discards the diamond, North discards a spade and the board is good; if West discards a spade, North discards the diamond. 7. East is now squeezed; if he discards the heart, South's heart is established and the ♠ A is the eighth trick. If East discards a spade, South will lead a low spade to North's ♠ A, establishing the final spade. 8. If, on the second trick, West discards a spade, North discards a low spade. The procedure is the same with the play of hearts and spades reversed. East plays a passive role in the hand — i.e., his best defense will not affect North/South play of the hand until the final squeeze.

Also solved by Rex Ingraham, Noland Poffenberger, Winslow H. Hartford, Michael A. Kay, William J. Butler, Jr., Paul W. Abrahams, Peter Wityk, Charles Polay, and the proposer, Emmet J. Duffy.

JAN 2 Consider two triangles ABC and PQR. Angle ABD = angle BDC = angle CDA = 120°. Prove that X = u + v + w. (The problem was published with X = u − v − w; since everyone who noticed that the problem was obviously false as printed also was able to deduce the correction, I will present the solution this month.) The following is from John F. Chandler:





Construct an equilateral triangle FGH with side u + v + w. Without loss of generality, we may assume u ≥ v ≥ w, as shown, and mark segments FG and GH with distances u, v, and w to locate points K, L, and M. Angle KGM is 60°, and KG = GM, so triangle KGM is also equilateral and KM = KG = v + w. Locate point N on KM and note that triangle KLN is also equilateral. Thus angle FKN = angle HMN = angle GLN = 120°. Now draw in segments GN, FN, and HN and observe that triangle FKN is congruent to triangle ABD, triangle HMN is congruent to triangle ADC, and triangle GLN is congruent to triangle DCB. Thus FN = c, HN = b, and GN = a. Note that the construction is unique in that the law of cosines provides unique solutions for angle FNG, angle FNH, and angle GNH in terms of a, b, and c. Thus triangle FGH is congruent to triangle PQR and x = u + v + w.

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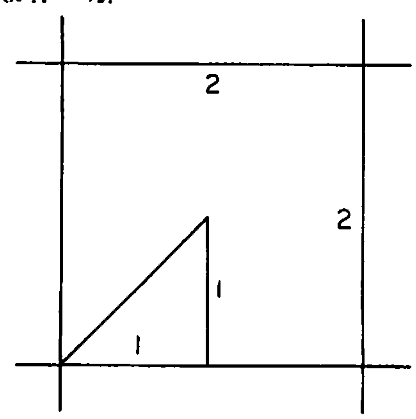
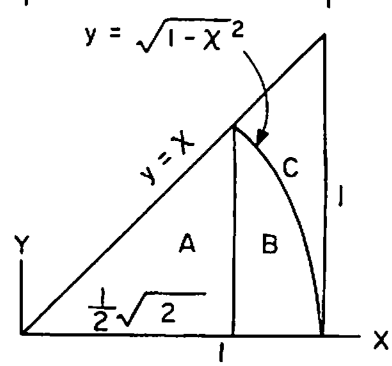
JAN 3 If you drop a six-inch pencil onto a tiled floor, each tile a 12-inch square, what is the probability that the pencil will cross at least one edge?

Several of the heavyweights responded and, surprisingly, there was some disagreement among their answers. Not surprisingly, I found R. Robinson Rowe's reasoning rock solid. His response follows:

The probability will be the same and the mathematics simpler if we use a six-inch pencil unit and drop a one-unit pencil on two-by-two-unit tiles. For each drop, take the tile in which the point of the pencil rests, with an equal probability of that point being anywhere on that tile, and an equal probability of the pencil's orientation thru a whole angle of 2 pi. From symmetry, the probability will be the same for any octant of the square and for the whole square, so I will compute it for the octant defined in the first diagram. It has an area of A = ½.





The second diagram shows this octant divided into three parts labeled A, B and C, with limits defined. Any random point will be defined as (x, y) from the axes shown. Let P be the probability for the tile and any octant. Let p be the probability for any point — varying from 0.75 at the origin to 0.0 at the center of the tile and 0.50 along the circular arc. For any point, p will be the ratio of the angle between intercepts on tile edges and the whole angle 2 pi.

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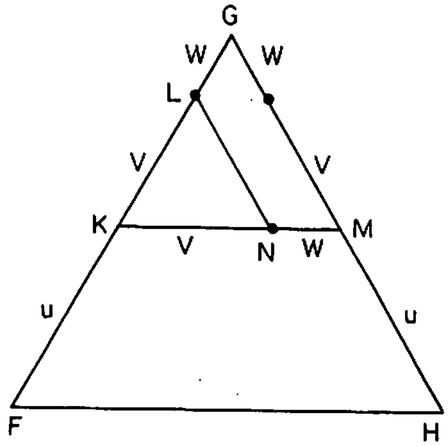

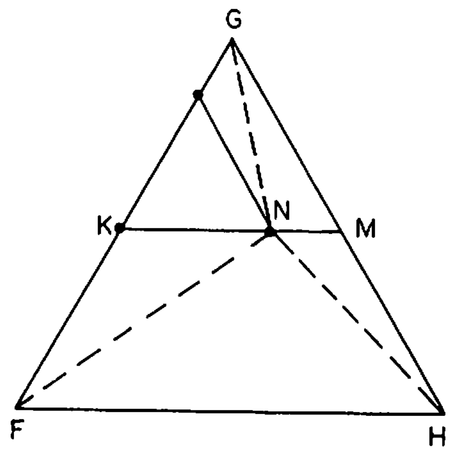

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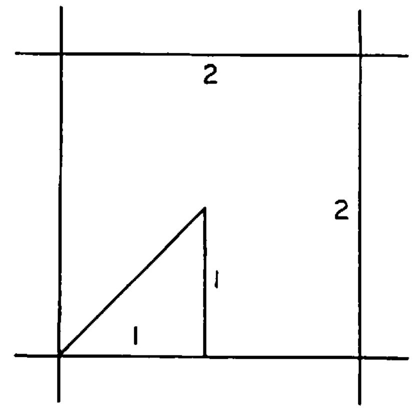

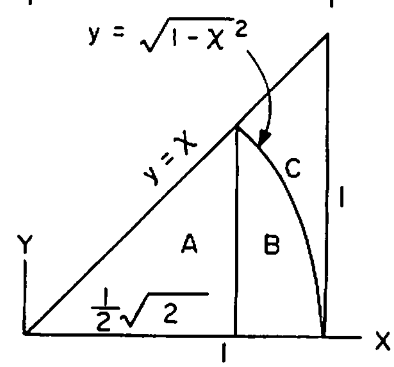

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Now let "tr" denote the trace of a matrix, which is just the sum of the elements on the main diagonal. tr A^i means the trace of the matrix A^i, using matrix multiplication to produce A^i and then taking the trace of the result. If A is an N × N matrix, prove that its determinant is the value of S_N.

Our only response is from Scott S Brown:

Reduction 1: We may as well assume that A is a matrix over C. We can do this because the polynomial relation det A = S_N can hold for all points in R^{N^2} only if it is a formal identity between the entries of A. Reduction 2: We may as well assume that A is a diagonal matrix. If A is not diagonal, then there is an invertable matrix T and a diagonal matrix A′ such that $A = TA'T^{-1}$. Now

$$\det A = (\det T)(\det A')(\det T^{-1}) = (\det A)(\det A')(\det T)^{-1} = \det A'.$$

And using tr AB = tr BA,

$$\text{tr } A = \text{tr}[T(A'T^{-1})] = \text{tr}(A'T^{-1}T) = \text{tr } A'.$$

Proof for $A = \text{diag}(a_1, a_2, \ldots, a_N)$, we have $A^n = \text{diag}(a_1^n, a_2^n, \ldots, a_N^n)$, so $\text{tr } A^n = \Sigma a_i^n$.

Let $\Delta(n, e) =$

$$\sum a_{i_1} a_{i_2} \ldots a_{i_{s-1}} a_{i_s}^e a_{i_{s+1}} \ldots a_{i_n}$$

where the sum is over

$1 \le i_1 < i_2 < \ldots < i_n \le N$ and

$1 \le s \le n$

Claim $S_n = \Delta(n, 1)$ for $n \ge 1$.

Proof by induction on n: For n = 1 the claim is clear. Assume that $S_k = \Delta(k, 1)$ for k = 1 to n − 1. Now

$$S_n = 1/n \cdot S_{n-1} \text{tr } A - S_{n-2} \text{tr } A^2 + \ldots + (-1)^{n-2} S_1 \text{tr } A^{n-1} + (-1)^{n-1}.$$ So $$\text{tr } A^n = 1/n \cdot \Delta(n-1, 1)(\Sigma a_i) - \Delta(n-2, 1)(\Sigma a_i^2) + \ldots + (-1)^{n-2} \Delta(1, 1) \Sigma a_i^{n-1} + (-1)^{n-1} \Sigma a_i^n = 1/n \cdot [n\Delta(n, 1) + \Delta(n-1, 2)] - [\Delta(n-1, 2) + \Delta(n-2, 3)] + \ldots + (-1)^{n-2}[\Delta(2, n-1) + \Delta(1, n)] + (-1)^{n-1}\Delta(1, n) = \Delta(n, 1),$$

which proves the claim. Taking n = N in the claim shows $S_n = a_1 a_2 \ldots a_n = \det A$ and proves the result.

Better Late Than Never

1975 M/A5 Professor R. L. Bishop has a *complete* solution which effectively supercedes the published version. He writes:

Since your published solutions are only fragmentary, I venture to point out that my own technique yields all solutions. If the desired positive integers a < b < c are added two at a time, let the sums be $A^2 < B^2 < C^2$. Further, let B = A + x and C = A + x + y. The general solution then calls for:

$$a = (A^2 - 2yA - y^2 - 2xy)/2$$
$$b = (A^2 + 2yA + y^2 + 2xy)/2$$
$$c = [A^2 + (2A + y)(2x + y) + 2x^2]/2$$

When one of these equations is satisfied by integers, subtraction proves that the others are also. Rewrite the first equation as:

$$A^2 - 2yA - y^2 = 2a + 2xy.$$

For each positive integer value of y there is an infinitely large family of solutions. It is required that y and A must be either both odd or both even. Secondly, for any given value of y and the minimum value of x = 1, it is necessary that

$$A > y + \sqrt{2}y(y + 1).$$

Finally, solutions then exist for all positive integer values of x, provided that

$$x < (A^2 - 2yA - y^2)/2y.$$

In contrast to the infinity of solutions for each value of y, your published solutions covered fully only the case of y = 1, partially the case of y = 2 (*only for x = 2)*, and other cases not at all except for the reference to multiplying by squares.

On a related topic (cf., J/A4) Emmet J. Duffy points out that to find N positive distinct integers such that the sum of any N − 1 integers is a perfect square, one need only find N integers a, b, c, etc., whose squares, a^2, b^2, c^2, etc., add up to a sum S which is exactly divisible by N − 1, and where S/(N − 1) is greater than any of the squares. The desired N distinct positive integers are then

$$S/(N-1) - a^2,\ S/(N-1) - b^2,\ S/(N-1) - c^2, \text{ etc.}$$

The integers, a, b, c, etc., can be consecutive numbers if N is a multiple of 6, or a multiple of 6 to which 2 has been added, and the smallest number, a, is $(N - 1)^2$.

JUN 4 Sidney Freidin, A. LeBlanc, and Avi Ornstein have responded.

O/N 1 Stanley G. Siegel has responded.

(Continued on p. 72)

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Puzzle

Continued from p. 29

O/N 4 L. J. Upton has pointed out that the solution given can be generalized to an arbitrary number of rivers.

Y1975 Ronald J. Brinkerhoff has responded, and Dr. Harry Hazard has found the following expressions with 1, 9, 7, and 5 in order:

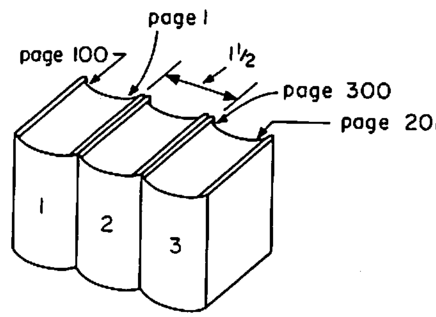
1	1 ** 975	44	(1 * 9) + (7 * 5)
5	(1 ** 97) * 5	45	1 + 9 + (7 * 5)
7	19 – 7 – 5	54	19 + (7 * 5)
11	(1 * 9) + 7 – 5	67	1 – 9 + 75
17	19 – 7 + 5	69	1 + (9 * 7) + 5
19	1 + (9 * [7 – 5])	75	(1 ** 9) * 75
21	19 + 7 – 5	81	(1 * 9) ** (7 – 5)
22	1 + 9 + 7 + 5	84	1 * (9 + 75)
27	1 – 9 + (7 * 5)	85	1 + 9 + 75
31	19 + 7 + 5	92	1 * (97 – 5)
32	1 * ([9 – 7] ** 5)	93	1 + 97 – 5
33	1 + ([9 – 7] ** 5)	94	19 + 75
36	(1 ** 9) + (7 * 5)		

NS1 Philip O. Martel has some partial results similar to those which appeared in 1966-67.

DEC 4 S. P. Hirshman has responded.

Proposers' Solutions to Speed Problems

SD1 Answer: 1.5 inches. The drawing shows that the worm eats through four covers and 100 pages.



SD2 The total distance from the first vessel to any point on shore and then to the second vessel would be the same if the second vessel were located nine miles inland. But if the second vessel were nine miles inland, the shortest distance would be a straight line which is $[(3 + 9)^2 + 5^2]^{\frac{1}{2}}$, or 13, miles. Thus the boat leaving the first vessel travels toward an imaginary vessel nine miles inland until it reaches shore and then travels to the other vessel for the minimum distance of 13 miles. (Note the similarity to O/N4. — Ed.)

Allan J. Gottlieb, who is Coordinator of Computer Activities at York College of the City University of New York, studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973). Send problems, solutions, and comments to him at York College, 150-14 Jamaica Avenue, Jamaica, N.Y., 11451.